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Bayesian spatial models for estimating means of sampled and non-sampled small areas

Hee Cheol Chung and Gauri S. Datta[1]

Abstract

In many applications, the population means of geographically adjacent small areas exhibit a spatial variation. If available auxiliary variables do not adequately account for the spatial pattern, the residual variation will be included in the random effects. As a result, the independent and identical distribution assumption on random effects of the Fay-Herriot model will fail. Furthermore, limited resources often prevent numerous sub-populations from being included in the sample, resulting in non-sampled small areas. The problem can be exacerbated for predicting means of non-sampled small areas using the above Fay-Herriot model as the predictions will be made based solely on the auxiliary variables. To address such inadequacy, we consider Bayesian spatial random-effect models that can accommodate multiple non-sampled areas. Under mild conditions, we establish the propriety of the posterior distributions for various spatial models for a useful class of improper prior densities on model parameters. The effectiveness of these spatial models is assessed based on simulated and real data. Specifically, we examine predictions of statewide four-person family median incomes based on the 1990 Current Population Survey and the 1980 Census for the United States of America.

Key Words: Conditional autoregression; Current Population Survey; Fay-Herriot model; Intrinsic autoregression; Mixed effects; Simultaneous autoregression; non-sampled small areas.

1. Introduction

Sample surveys provide useful data in estimating various characteristics of a population of interest. Surveys are generally designed so that design-based estimators have adequate accuracy. However, when it comes to estimating a sub-population characteristic, a design-based direct estimate, based solely on data from that sub-population alone, is usually inaccurate as the accessible sample size is small and sometimes nonexistent. Sub-populations that lack a reasonable sample size to produce reliable direct estimates are known as small areas. Also, limited resources often preclude many sub-populations from selecting in the sample, creating non-sampled small areas. For example, the American Community Survey (ACS) is conducted to produce reliable statistics for the U.S. counties. However, the ACS usually samples about one-third of the counties resulting in many non-sampled small areas.

To enhance the accuracy of direct estimates of small areas, a model-based approach has been widely used to facilitate borrowing information from direct estimates of other domains and other auxiliary data. In many applications, supplementary information from other surveys and administrative data provide useful covariates. A model-based estimate of an area is produced by suitably shrinking its direct estimate (if available) to a synthetic regression estimate based on auxiliary variables. The improvement in prediction greatly depends on to what extent the sub-population means of the characteristic are related to the auxiliary variables. If a small area has no direct estimate, the traditional independent random-effects model of Fay and Herriot (1979) estimates estimates the mean by a synthetic regression estimate alone.

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Fay and Herriot (1979) proposed a useful model for developing estimates of small area means based on direct survey estimates (if available) and computed synthetic regression estimates from auxiliary variables. This model, which is essentially a mixed linear model, is popularly known as the Fay-Herriot (FH) model in small area estimation. For $i = 1, \ldots, m,$ let Y_i be the direct estimate of the small area characteristic θ_i obtained from a survey. Also let $\mathbf{x}_i$ and $\boldsymbol{\beta}$ be the p-component vectors of covariates and corresponding regression coefficients, respectively. Denoting the sampling error of Y_i as e_i, the independent FH model can be written as

$$Y_i = \theta_i + e_i, \quad \theta_i = \mathbf{x}_i^{\mathrm{T}} \boldsymbol{\beta} + v_i, \quad i = 1, \ldots, m, \tag{1.1}$$

where e_i's and random effects v_i's $i = 1, \ldots, m,$ are all independently distributed with $e_i \sim N(0, D_i),$ and $v_i \overset{\text{i.i.d.}}{\sim} N(0, \sigma_v^2).$ Sampling variances D_i, $i = 1, \ldots, m$ are taken as known, whereas the regression parameter β and model error variance σ_v^2, called model parameters, are unknown quantities. For non-sampled areas with auxiliary variables, only the second part of (1.1) holds for θ_i.

There has been extensive research on the independent FH model and its many variants. While Fay and Herriot (1979) used an empirical Bayes (EB) approach, subsequently, Prasad and Rao (1990), Datta and Lahiri (2000) and Datta, Rao and Smith (2005) used the frequentist approach and derived the second-order mean squared error (MSE) of empirical best linear unbiased predictor (EBLUP) of θ_i and various second-order approximate unbiased estimators of the MSE's (see Datta and Lahiri, 2000). However, Ghosh (1992) proposed a hierarchical Bayesian (HB) approach for the Fay-Herriot model (see also Datta et al. (2005)). In the Bayesian framework, the FH model in (1.1) can be expressed as the following HB model:

$$Y_i \,|\, \theta_1, \ldots, \theta_m, \boldsymbol{\beta}, \sigma_v^2 \overset{\text{ind}}{\sim} N(\theta_i, D_i), \quad i = 1, \ldots, m, \tag{1.2}$$

$$\theta_i \,|\, \boldsymbol{\beta}, \sigma_v^2 \overset{\text{ind}}{\sim} N(\mathbf{x}_i^{\mathrm{T}} \boldsymbol{\beta}, \sigma_v^2), \quad i = 1, \ldots, m, \tag{1.3}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2) \propto g(\boldsymbol{\beta}, \sigma_v^2), \tag{1.4}$$

where $g(\cdot)$ is a suitably chosen function of $\boldsymbol{\beta}$ and σ_v^2, which expresses a prior probability density function (pdf) for these parameter. An EB predictor for θ_i, which does not require a prior pdf as in (1.4), was originally developed by Fay and Herriot (1979). While a standard EB approach usually underestimates the measure of uncertainty of the EB estimator of θ_i, the HB approach facilitates quantification of uncertainty due to estimation of unknown model parameters, $\boldsymbol{\beta}$ and σ_v^2. The uncertainty is fully captured by the posterior distribution of the model parameters.

In model-based estimations, random effects are of great importance in capturing the remaining variability of the θ_i's that is not explained by the regression model. In real applications, small areas generally involve features such as population size, ethnicity, age-group, and education level, which might affect the variability of small area effects. Furthermore, when disease prevalence rates are of interest, it is reasonable to assume that random effects of adjacent small areas are correlated in a certain way. In such cases, the FH model given in (1.1), which we refer to as the independent FH random-effects model, oversimplifies and misspecifies the distribution of random effects by assuming a common and independent distribution. Opsomer, Claeskens, Ranalli, Kauermann and Breidt (2008) and Rao, Sinha and Dumitrescu (2014) proposed nonparametric small area estimation models, which capture spatial proximity effect using the P-spline function. However, these approaches require additional computational cost for model inference and uncertainty quantification.

In this work, we propose spatial FH models which effectively account for heteroscedasticity and spatial dependence of the small area effects. We take a fully Bayesian approach by specifying a class of noninformative priors on the model parameters and model spatial dependence of small area random effects by four widely used autocorrelation structures. These include simultaneous autoregressive and three types of conditional autoregressive models. There is an abundance of literature on spatial models under the Bayesian framework. Sun, Tsutakawa and Speckman (1999) studied an HB model with the conditional and intrinsic autoregressive models on the random effects. The same models were considered by Speckman and Sun (2003) in the context of Bayesian spline smoothing. For small area estimation, You and Zhou (2011) modeled small area effects using a conditional autoregressive model. As an extension of the time series FH model (Datta, Lahiri, Maiti and Lu, 1999), Torabi (2012) proposed a spatio-temporal model with intrinsic autoregressive random effects. Porter, Holan, Wikle and Cressie (2014) proposed an extension of the FH model with functional covariates and intrinsic autoregressive random effects. Porter, Wikle and Holan (2015) incorporated the conditional autoregressive random effects on the multivariate FH model.

The existing Bayesian spatial small area estimation models consider a proper prior on σ_v^2 even though the specification of a proper prior will require subject matter expertise. Furthermore, all existing models assume a conditional autoregressive structure on the random effects. The main contributions of this paper are as follows. First, to the best of our knowledge, the proposed models in Section 2 (Section 2.1) include most of the popularly used spatial structures. Second, in Section 2.2, we further extend the spatial models to estimate means of several non-sampled small areas with no direct estimates. The non-sampled area mean θ_i is estimated by borrowing strength from the auxiliary variables of the area and, for spatial models, from the regression residuals of its neighboring areas. Third, for all proposed models, we provide, in Section 2.3, sufficient conditions for posterior propriety for a class of improper noninformative priors on model parameters. Interestingly, the sufficient conditions do not depend on the assumed spatial model, provided that the model yields a positive definite covariance matrix for the random effects. We provide rejection sampling steps for simulating from the posterior of the proposed models in Section 3. The effectiveness of the proposed spatial models is demonstrated in Sections 4 and 5. We apply the spatial models to simulated datasets and real survey data from the Current Population Survey (CPS). We compare various spatial models in Section 5 to estimate four-person family median incomes for the forty-nine contiguous states of the U.S. based on the CPS data and appropriate covariates from the previous Census and administrative data. Our data analysis and simulation studies reveal that proposed spatial models significantly improve prediction accuracy and reduce the measure of uncertainty, posterior standard deviation. We provide concluding remarks in Section 6. All technical details are provided in Appendix.

2. Some spatial alternatives to the independent FH model

2.1 Incorporating spatial random effects

Let $\mathbf{Y} = (Y_1, \ldots, Y_m)^\top$ be the m-component vector with the direct estimates of m small areas, and $\mathbf{D} = \operatorname{diag}\{D_i\}_{i=1}^m$ be the $m \times m$ diagonal matrix with the sampling variances of the direct estimates. We denote by $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)^\top$ the m-component vector of small area means. Also, let $\mathbf{x}_i \in \mathbb{R}^p$ be the p-component vector of auxiliary variables (including the intercept term) for the i^{th} small area, and $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_m]^\top$. A special case of the HB model given in (1.2)-(1.4) can be expressed as

$$\mathbf{Y} \mid \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2 \sim N_m(\boldsymbol{\theta}, \mathbf{D}), \tag{2.1}$$

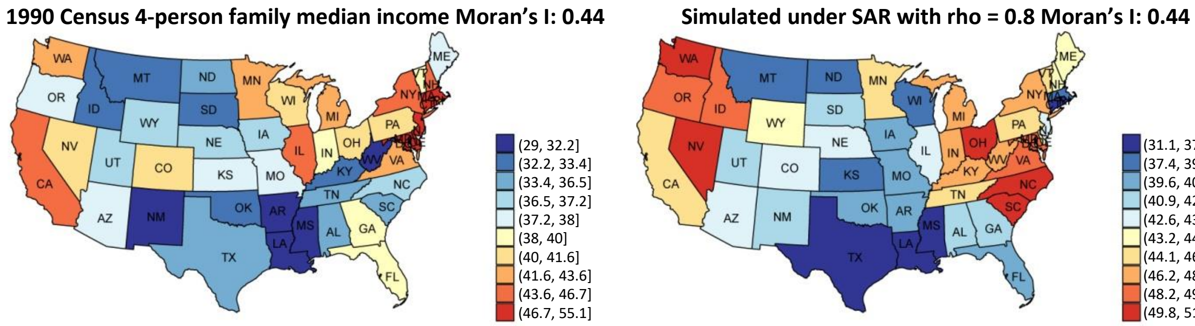
$$\boldsymbol{\theta} \mid \boldsymbol{\beta}, \sigma_v^2 \sim N_m(\mathbf{X}\boldsymbol{\beta}, \sigma_v^2 \mathbf{I}_m), \tag{2.2}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2) \propto 1, \tag{2.3}$$

where $\boldsymbol{\beta}$ is the p-component regression coefficient vector, σ_v^2 is the model error variance, and $\mathbf{I}_m$ is the identity matrix of order m. The uniform prior (2.3) on the model parameters is a popularly used noninformative prior, and the resulting posterior pdf is proper provided that $m > p + 2$. See Berger (1985) and Datta and Smith (2003) for detailed discussion.

The model (2.2) assumes that θ_i, $i = 1, \ldots, m$, are independently distributed over the small areas with common random effects variance σ_v^2. In many small area estimation problems, however, the area characteristic of interest is closely related to geographical factors such as population size, ethnicity, age-group and education level. When available covariates do not fully explain such spatial association, the independence and equal variance assumptions of random effects fail, and inference based on the hierarchical model (2.1)-(2.3) may generate unreliable estimates, consequently resulting in erroneous decisions. Figure 2.1 illustrates spatial associations of the 1990 Census 4-person family median incomes of (scaled by \$1,000) the $m = 49$ states of the U.S., including the District of Columbia. Simulated data with the same Moran's I value are also displayed for comparison, where Moran's I is a measure for spatial autocorrelation. The simulated data are generated under the SAR model (defined below) with $\rho = 0.8$ matching the location and scale with the Census data. The two panels demonstrate the existence of spatial dependence in the 1990 Census 4-person family median incomes. In practice, covariates capable of fully capturing existing spatial variation are not always available, and the problem can be exacerbated if lurking variables exist, as they introduce additional variability that cannot be explained by the independently and identically distributed (i.i.d.) random effects.

Figure 2.1 Graphical illustrations of the 1990 Census 4-person family median incomes of the $m = 49$ states of U.S. and simulated data.



- The simulated data are generated under the SAR model (defined below) with $\rho =$ 0.8 and have the same Moran's I estimate.
- AL = Alabama, AZ = Arizona, AR = Arkansas, CA = California, CO = Colorado, CT = Connecticut, DE = Delaware, DC = District of Columbia, FL = Florida, GA = Georgia, ID = Idaho, IL = Illinois, IN = Indiana, IA = Iowa, KS = Kansas, KY = Kentucky, LA = Louisiana, ME = Maine, MD = Maryland, MA = Massachusetts, MI = Michigan, MN = Minnesota, MS = Mississippi, MO = Missouri, MT = Montana, NE = Nebraska, NV = Nevada, NH = New Hampshire, NJ = New Jersey, NM = New Mexico, NY = New York, NC = North Carolina, ND = North Dakota, OH = Ohio, OK = Oklahoma, OR = Oregon, PA = Pennsylvania, RI = Rhode Island, SC = South Carolina, SD = South Dakota, TN = Tennessee, TX = Texas, UT = Utah, VT = Vermont, VA = Virginia, WA = Washington, WV = West Virginia, WI = Wisconsin, WY = Wyoming.

To address this issue, we propose to use spatially correlated random effects. Let $\mathbf{W} = \{w_{ij}\}_{ij}$, $1 \le i, j \le m,$ be the adjacency matrix which plays an important role in capturing spatial dependency. In particular, $w_{ij} = 1$ if the i^{th} and j^{th} small areas are connected via geographical boundary or through other mechanisms (for example, air traffic), and $w_{ij} = 0,$ otherwise. Also, $w_{ii} = 0$ for $i = 1, \ldots, m.$ The off-diagonal entries, w_{ij}'s, need not be binary; they can take other positive values, such as the "length" of the geographical border or volumes of air traffic between the two areas. Since the adjacency matrix $\mathbf{W}$ is symmetric, its eigenvalues are real. We denote the i^{th} largest eigenvalue of $\mathbf{W}$ by $\lambda_i(\mathbf{W}),$ such that $\lambda_m(\mathbf{W}) \le \ldots \le \lambda_1(\mathbf{W}).$ Since $\mathbf{W}$ is non-null and $\sum_{i=1}^m w_{ii} = 0,$ we get as a result that $\lambda_m(\mathbf{W}) < 0 < \lambda_1(\mathbf{W}).$ Let $w_{i.} = \sum_{j=1}^m w_{ij}$ be the sum of the i^{th} row of $\mathbf{W}$ and $\mathbf{L} = \operatorname{diag}\{w_{i.}\}_{i=1}^m.$ Assuming that diagonal elements of $\mathbf{L}$ are positive, i.e., all small areas have at least 1 neighboring area, we define $\tilde{\mathbf{W}} = \mathbf{L}^{-1}\mathbf{W}.$ Since $\tilde{\mathbf{W}}$ is a row stochastic matrix, all of its eigenvalues are between -1 and 1, with at least one of them is 1. Consequently, $\lambda_1(\tilde{\mathbf{W}}) = 1.$ Moreover, $\tilde{\mathbf{W}}$ and $\operatorname{diag}\{w_{i.}^{-1/2}\}_{i=1}^m \mathbf{W} \operatorname{diag}\{w_{i.}^{-1/2}\}_{i=1}^m$ have the same set of eigenvalues, and the latter matrix is symmetric. So all the eigenvalues of $\tilde{\mathbf{W}}$ are real, and $\lambda_m(\tilde{\mathbf{W}})$ will be negative. We consider four alternative spatial dependencies associated with random effects, which are represented by the following positive definite precision matrices (excluding the scale parameter σ_v^2):

$$\text{SAR}: \quad \mathbf{\Omega}_2(\rho) = (\mathbf{I}_m - \rho\tilde{\mathbf{W}})^\top (\mathbf{I}_m - \rho\tilde{\mathbf{W}}), \quad \rho \in (-1, 1), \tag{2.4}$$

$$\text{SCAR}: \quad \mathbf{\Omega}_3(\rho) = \mathbf{I}_m - \rho\mathbf{W}, \quad \rho \in (\lambda_m(\mathbf{W})^{-1}, \lambda_1(\mathbf{W})^{-1}), \tag{2.5}$$

$$\text{CAR}: \quad \mathbf{\Omega}_4(\rho) = \mathbf{L} - \rho\mathbf{W}, \quad \rho \in (\lambda_m(\tilde{\mathbf{W}})^{-1}, \lambda_1(\tilde{\mathbf{W}})^{-1}), \tag{2.6}$$

$$\text{LCAR}: \quad \mathbf{\Omega}_5(\rho) = \rho\mathbf{R} + (1-\rho)\mathbf{I}_m, \quad \rho \in (0, 1), \tag{2.7}$$

where ρ is the spatial dependence parameter that represents the strength of spatial dependence (Hodges, 2019, Chapter 5.2) and $\mathbf{R}$ is defined as $\mathbf{R} = \mathbf{\Omega}_4(1) = \mathbf{L} - \mathbf{W}.$ Since the eigenvalues of $\mathbf{I}_m - \tilde{\mathbf{W}}$ are between 0 (the smallest eigenvalue) and $1-\lambda_m(\tilde{\mathbf{W}})$ (the largest eigenvalue, >1), the matrix $\mathbf{R}$ is nonnegative definite. Each precision matrix is guaranteed to be positive definite as long as ρ is in the range specified in the respective definition.

The adjacency matrix $\tilde{\mathbf{W}}$ of the simultaneous autoregressive (SAR) model (Whittle, 1954) is row-normalized so that ρ can vary from -1 to 1 while preserving the positive definiteness (Banerjee, Carlin and Gelfand, 2003, Chapter 4.4). The model (2.5) is a simple version of conditional autoregressive (CAR) model (Rao and Molina, 2015, Chapter 9.6.2), where diagonal entries of the precision matrix are all equal to one. Even though the diagonal elements of a precision matrix are all equal, the diagonal elements of the inverse may not be all equal, leading to heteroscedasticity of random effects. We call this model the simple conditional autoregressive (SCAR) model. The model (2.6) is widely used conditional autoregressive model (CAR; Banerjee et al., 2003; Besag and Kooperberg, 1995; You and Zhou, 2011), where diagonal entries of the precision matrix are the number of neighborhoods of the corresponding area. The upper limit of ρ is $\lambda_1(\tilde{\mathbf{W}})^{-1}=1$, and in the case of $\rho=1$, the model with $\mathbf{\Omega}_4(1)$ is referred to as the intrinsic autoregressive (IAR) model (Banerjee et al., 2003, Chapter 4.3). The model (2.7) is a conditional autoregressive model, which we call Leroux's conditional autoregressive (LCAR), whose precision matrix is given by the convex combination of $\mathbf{R}=\mathbf{\Omega}_4(1)$ and $\mathbf{I}_m$. This model has been considered by Leroux, Lei and Breslow (2000); MacNab (2003); You and Zhou (2011), where the i^{th} diagonal element of $\mathbf{R}$ is the number of neighborhoods of the i^{th} small area, and the $(i,j)^{\text{th}}$ off-diagonal element is -1 if the i^{th} and the j^{th} small areas are connected and 0 otherwise.

The conditional autoregressive models, SCAR, CAR, and LCAR, assume that θ_i depends only on neighboring small area means. In other words, θ_i is correlated with θ_j's, $j\neq i$, only through the means of surrounding areas. On the contrary, the SAR model assumes that θ_i is dependent on all other θ_j concurrently, $j\neq i$, but has stronger (weaker) correlations for neighboring (remote) areas. The independent FH model can be viewed as a special case of the SAR, SCAR, or LCAR model with $\rho=0$. For notational convenience, we include the independent FH model as part of our model by taking its precision matrix $\mathbf{\Omega}_1(\rho)=\mathbf{I}_m$, although it is free from ρ.

We consider the following HB spatial models incorporating the five spatial dependencies defined in (2.4)-(2.7):

$$\mathbf{Y}\,|\,\boldsymbol{\theta},\boldsymbol{\beta},\sigma_v^2,\rho \sim N_m(\boldsymbol{\theta},\mathbf{D}), \tag{2.8}$$

$$\boldsymbol{\theta}\,|\,\boldsymbol{\beta},\sigma_v^2,\rho \sim N_m(\mathbf{X}\boldsymbol{\beta},\sigma_v^2\{\mathbf{\Omega}_k(\rho)\}^{-1}),\quad k=1,\ldots,5, \tag{2.9}$$

$$\pi(\boldsymbol{\beta},\sigma_v^2,\rho) \propto g(\sigma_v^2)\,h(\rho),\quad \boldsymbol{\beta}\in\mathbb{R}^p,\sigma_v^2>0,\ l_k<\rho<u_k, \tag{2.10}$$

where σ_v^2 is the model scale parameter, $g(\sigma_v^2)$ and $h(\rho)$ are suitable functions of σ_v^2 and ρ, l_k and u_k are the lower and upper bounds of ρ under the k^{th} model. We avoid the term "model error variance" for σ_v^2 as diagonal entries of $\mathbf{\Omega}_k(\rho)$ vary across small areas and are not necessarily all one.

2.2 Estimation of population means of non-sampled small areas

In this section, we consider the case when, in the survey, there are several non-sampled small areas that have no direct estimates. In many applications, limited resources frequently preclude the inclusion of many subpopulations in the sample, resulting in non-sampled small areas. In this section, we consider the case when, in the survey, there are several non-sampled small areas that have no direct estimates. In many applications, limited resources frequently preclude the inclusion of many subpopulations in the sample, resulting in non-sampled small areas. Non-sampled small areas are sometimes referred to as misaligned areas (Trevisani and Gelfand, 2013) when they arise from domain misalignment between the direct estimate and auxiliary variables. For any of these non-sampled areas, the prediction of its mean from any non-spatial model is only based on its synthetic estimator. We propose to exploit spatial dependencies in predicting area means of non-sampled small areas. The predictions of the proposed models are obtained by modifying its synthetic estimator, using the vector of regression residuals, with more emphasis on the regression residuals of the neighboring areas.

Without loss of generality, let there be m_1 non-sampled small areas and $Y_{m_1+1}, \ldots, Y_m$ be the direct estimates of the $m_2 = m - m_1$ sampled small areas. Based on the direct estimates of m_2 sampled areas, we consider the following HB models:

$$\mathbf{Y}_{(2)} \,\big|\, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho \ \sim\ N_{m_2}(\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)}), \tag{2.11}$$

$$\boldsymbol{\theta} \,|\, \boldsymbol{\beta}, \sigma_v^2, \rho \ \sim\ N_m\left(\mathbf{X}\boldsymbol{\beta}, \sigma_v^2 \left\{\boldsymbol{\Omega}_k(\rho)\right\}^{-1}\right), \quad k = 1, \ldots, 5, \tag{2.12}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho) \ \propto\ g(\sigma_v^2)\, h(\rho), \quad \boldsymbol{\beta} \in \mathbb{R}^p, \sigma_v^2 > 0,\ l_k < \rho < u_k, \tag{2.13}$$

where $\mathbf{Y}_{(2)} = (Y_{m_1+1}, \ldots, Y_m)^\top$, $\mathbf{D}_{(2)} = \operatorname{diag}\{D_i\}_{i=m_1+1}^m$, and $\boldsymbol{\theta}_{(2)} = (\theta_{m_1+1}, \ldots, \theta_m)^\top$, which is the subvector of $\boldsymbol{\theta}$ corresponding to the sampled areas.

2.3 Propriety of the posterior distributions

In this section, we establish propriety of the posterior distributions of spatial small area models given in (2.8)-(2.10) and (2.11)-(2.13). Let $I(\cdot)$ be the indicator function taking the value 1 when its argument is true and 0 otherwise. We first provide general conditions for the posterior propriety of the proposed models.

Theorem 1. For all the HB spatial models given in (2.8)-(2.10) and (2.11)-(2.13), the posterior probability density functions are proper if the following conditions hold for some positive constant $c > 0$:

(a) $\int_0^\infty g(\sigma_v^2)\, I(\sigma_v^2 \le c)\, d\sigma_v^2 < \infty.$

(b) $\int_0^\infty (\sigma_v^2)^{-(m^* - p)/2}\, g(\sigma_v^2)\, I(\sigma_v^2 > c)\, d\sigma_v^2 < \infty.$

(c) $\int_{l_k}^{u_k} h(\rho)\, d\rho < \infty,$

where $m^* = m$ for (2.8)-(2.10), and $m^* = m - m_1$ for (2.11)-(2.13).

If $g(\cdot)$ is a proper pdf, then (a) holds true automatically, and (b) is satisfied if $m^* \geq p.$ The condition $m^* \geq p$ is obvious since at least p observations are needed to estimate p components of $\boldsymbol{\beta}$ when no substantive information about it is available. Also, any bounded function of ρ satisfies (c) in Theorem 1 as their supports are all bounded. In particular, under the popular family of noninformative priors

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho) \propto (\sigma_v^2)^{-\alpha} I(l_k < \rho < u_k), \quad \boldsymbol{\beta} \in \mathbb{R}^p, \sigma_v^2 > 0, \tag{2.14}$$

the posterior pdfs are proper under the following conditions.

Corollary 1. For any of the HB spatial models given in (2.8)-(2.9) and (2.11)-(2.12) with the prior in (2.14), the posterior pdf is proper as long as $\alpha < 1$ and $m^* > p + 2 - 2\alpha.$

For the uniform prior with $\alpha = 0$ (which will be used in this paper), the propriety of the posterior distributions for models (2.8)-(2.9) are guaranteed as long as the number of small areas is greater than $p + 2.$ For the models incorporating non-sampled areas given in (2.11)-(2.12), the second condition of Corollary 1.1 becomes $m - m_1 > p + 2,$ and thus, the posterior pdfs are proper as long as the number of non-sampled areas is fewer than $m - p - 2,$ or at least $p + 3$ areas have sample.

3. Simulating posterior distributions

In this section, we illustrate the rejection sampling steps to obtain independent posterior samples from the posterior distributions of proposed models. We assume that the components of the small area mean vector $\boldsymbol{\theta}$ are arranged so that $\boldsymbol{\theta} = (\boldsymbol{\theta}_{(1)}^\top, \boldsymbol{\theta}_{(2)}^\top)^\top,$ where $\boldsymbol{\theta}_{(1)} \in \mathbb{R}^{m_1}$ and $\boldsymbol{\theta}_{(2)} \in \mathbb{R}^{m_2}$ are the small area mean vectors corresponding to the non-sampled and sampled areas, respectively. For notational convenience, we denote the precision matrix of the k^{th} spatial model by $\boldsymbol{\Omega} = (\sigma_v^2)^{-1} \boldsymbol{\Omega}_k(\rho)$ and the permissible range of ρ by (l, u) suppressing the model index $k.$

We first derive the marginal posterior density of (σ_v^2, ρ) and provide subsequent sampling procedures. Let $\mathbf{0}_{m_2 \times m_1}$ be the $m_2 \times m_1$ null matrix and $\mathbf{M} = [\mathbf{0}_{m_2 \times m_1}, \mathbf{I}_{m_2}]$ such that $\boldsymbol{\theta}_{(2)} = \mathbf{M}\boldsymbol{\theta}.$ We also let $\mathbf{X}_{(2)} = \mathbf{MX}.$ Integrating out $\boldsymbol{\theta}$ from the model (2.11)-(2.12), we have $\mathbf{Y}_{(2)} \,|\, \boldsymbol{\beta}, \sigma_v^2, \rho \sim N_{m_2}(\mathbf{X}_{(2)} \boldsymbol{\beta}, \boldsymbol{\Delta}),$ where $\boldsymbol{\Delta} = \mathbf{D}_{(2)} + \mathbf{M}\boldsymbol{\Omega}^{-1}\mathbf{M}^\top.$ Subsequent marginalization of $\boldsymbol{\beta}$ gives the marginal posterior density $p(\sigma_v^2, \rho \,|\, \mathbf{y}_{(2)})$ as

$$p(\sigma_v^2, \rho \,|\, \mathbf{y}_{(2)}) \propto \frac{\exp\left[-\frac{1}{2} \mathbf{y}_{(2)}^\top \boldsymbol{\Delta}^{-1} \left\{\boldsymbol{\Delta} - \mathbf{X}_{(2)} (\mathbf{X}_{(2)}^\top \boldsymbol{\Delta}^{-1} \mathbf{X}_{(2)})^{-1} \mathbf{X}_{(2)}^\top\right\} \boldsymbol{\Delta}^{-1} \mathbf{y}_{(2)}\right]}{|\boldsymbol{\Delta}|^{1/2} \, |\mathbf{X}_{(2)}^\top \boldsymbol{\Delta}^{-1} \mathbf{X}_{(2)}|^{1/2}} I(l < \rho < u). \tag{3.1}$$

Furthermore, we have conditional posterior distributions of $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$ as

$$\boldsymbol{\beta} \,|\, \sigma_v^2, \rho, \mathbf{y} \sim N_p(\boldsymbol{\gamma}, \boldsymbol{\Gamma}), \tag{3.2}$$

$$\boldsymbol{\theta} \,|\, \boldsymbol{\beta}, \sigma_v^2, \rho, \mathbf{y} \sim N_m(\boldsymbol{\mu}, \boldsymbol{\Psi}), \tag{3.3}$$

where $\mathbf{\Gamma} = (\mathbf{X}_{(2)}^{\top} \mathbf{\Delta}^{-1} \mathbf{X}_{(2)})^{-1}$, $\boldsymbol{\gamma} = \mathbf{\Gamma}\mathbf{X}_{(2)}^{\top}\mathbf{\Delta}^{-1}\mathbf{y}_{(2)}$, $\boldsymbol{\mu} = \mathbf{y}_{*} - \mathbf{\Psi}\mathbf{\Omega}(\mathbf{y}_{*} - \mathbf{X}\boldsymbol{\beta})$, $\mathbf{y}_{*} = (\mathbf{0}_{m_1}^{\top}, \mathbf{y}_{(2)}^{\top})^{\top}$, and $\mathbf{\Psi}^{-1} = \mathbf{M}^{\top}\mathbf{D}_{(2)}^{-1}\mathbf{M} + \mathbf{\Omega}$. Accordingly, we can obtain a independent posterior sample via rejection sampling from (3.1) and subsequent samplings from (3.2) and (3.3). For the data with no non-sampled area, we have desired sampling procedures by setting $\mathbf{M} = \mathbf{I}_m$ and $\mathbf{y}_{*} = \mathbf{y}$.

4. A simulation study

In this section, we compare prediction performances of the independent FH model and the four spatial models in the absence of informative covariates with multiple non-sampled areas. Excluding Hawaii and Alaska, we consider contiguous $m = 49$ states of the U.S., including the District of Columbia. To evaluate the quality of prediction in the absence of direct estimates, we do not simulate direct estimates of randomly chosen $m_1 = \lfloor 0.15m \rfloor = 7$ states. These areas are Delaware, Massachusetts, Michigan, Nebraska, Rhode Island, South Dakota, and Texas. This results in $m_2 = 42$ areas which have direct estimates.

To make simulation settings realistic, we mimic the 1989 4-person family median income (median income) data described in Section 5. We generate replicated datasets so that Moran's I values of each replicated small area means, $\theta_1, \ldots, \theta_m$, are approximately centered around 0.44, the Moran's I value of the 1990 Census median income. Direct estimates are generated with the sampling variances $D_1, \ldots, D_{m_2}$ of the 1990 Current Population Survey (CPS) estimates. These sampling variances of sampled small areas range from 1.95 to 25.03 with the mean of 9.08, where dollar amounts are scaled by \$1,000. For each setting, we consider $S = 100$ replicated datasets.

Data generation: Let $\bar{D} = m_2^{-1} \sum_{i=1}^{m_2} D_i$ and $\mathbf{D}_{(2)} = \mathrm{diag}\{D_i\}_{i=1}^{m_2}$. We set $\rho = 0.85$ and $\sigma_v^2 = \bar{D}/2$ and consider two independent covariates $\mathbf{x}_1$ and $\mathbf{x}_2$ with SAR spatial dependence, i.e., $\mathbf{x}_1, \mathbf{x}_2 \sim N_m\left(\mathbf{0}_m, \{\mathbf{\Omega}_3(\rho)\}^{-1}\right)$. Then, letting $(\beta_1, \beta_2)^{\top} = (2, 1)^{\top}$ and $\boldsymbol{\mu} = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2$, we generate small area means and direct estimates from the following independent FH model:

$$\boldsymbol{\theta} \sim N_m(\boldsymbol{\mu}, \sigma_v^2 \mathbf{I}_m), \quad \mathbf{Y}_{(2)} \big| \boldsymbol{\theta}_{(2)} \sim N_{m_2}(\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)}),$$

where the components of $\mathbf{Y}_{(2)}$ and $\boldsymbol{\theta}_{(2)}$ correspond to the m_2 sampled small areas as defined below equation (2.13). The covariate $\mathbf{x}_1(\mathbf{x}_2)$ introduces stronger (weaker) spatial pattern to the θ_i's, and accordingly, we call $\mathbf{x}_1$ $(\mathbf{x}_2)$ as the strong (weak) covariate. Moran's I values of 100 replicated small area means range from 0.115 to 0.713 with mean 0.449.

We consider two different covariate settings to examine how the spatial models can capture extra variability introduced by the spatial dependence from a missing covariate, i.e., $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_2]$ and $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_1]$, where $\mathbf{1}_m$ represents the m-component vector of ones. Excluding any of the covariates from the fitted model will leave the spatial variation of that covariate to the residual. We do not consider the full model involving both the covariates since that model will fully capture $\boldsymbol{\mu}$ and leave no spatial variability unexplained; consequently, the independent FH model will be sufficient to capture the variability of the i.i.d. random effects.

Posterior simulations: For all models proposed, independent posterior samples can be obtained by the rejection sampling scheme outlined in Section 3. The sampling procedure begins with the rejection sampling from the marginal posterior distribution of (σ_v^2, ρ) and continues with successive samplings of the rest of the parameters from the conditional posterior distributions. However, when the marginal posterior density of (σ_v^2, ρ) is concentrated at or around the boundaries of $\rho,$ a proposal distribution must be carefully specified to have a sufficiently high acceptance rate, which may require adaptive specification a proposal distribution for each replicated dataset. To avoid such difficulties, we use Hamiltonian Monte Carlo algorithm with the `R` package `rstan` (Stan Development Team, 2018). We fit the HB model (2.11)-(2.13) with each combination of covariates for $k = 1, \ldots, 5.$ For each model, we run four parallel Hamiltonian Monte Carlo chains (No-U-Turn Sampler) for 2,500 iterations after 5,000 burn-in iterations. We keep every 10$^{\text{th}}$ iteration and concatenate the four chains to obtain a posterior sample of size 10,000. The `R` codes implementing the rejection sampling step described in Section 3 and the `stan` models are available at https://github.com/heech31/spatial_sae.

Measures of performance: With the posterior sample under each model, we predict the true small area mean vector, $\boldsymbol{\theta}^{(s)} = (\theta_1^{(s)}, \ldots, \theta_m^{(s)})^{\top},$ of the s^{th} replicated dataset using the posterior mean, which we denote by $\hat{\boldsymbol{\theta}}^{(s)} = (\hat{\theta}_1^{(s)}, \ldots, \hat{\theta}_m^{(s)})^{\top}.$ Let $\mathcal{A}$ be a subset of $\{1, \ldots, m\},$ which is determined by indices only of the sampled or non-sampled small areas. For a given subset $\mathcal{A},$ we calculate the mean squared prediction error, $\text{MSPE}^{(s)} = \sum_{i \in \mathcal{A}} (\hat{\theta}_i^{(s)} - \theta_i^{(s)})^2 \big/ m_{\mathcal{A}},$ where $m_{\mathcal{A}} = |\mathcal{A}|$ is the number of areas in $\mathcal{A}.$ We then average $\text{MSPE}^{(s)}$ over S replications to compute the average empirical mean squared prediction error (AeMSPE), where

$$\text{AeMSPE} = \frac{1}{S} \sum_{s=1}^{S} \text{MSPE}^{(s)} = \frac{1}{S} \sum_{s=1}^{S} \frac{1}{m_{\mathcal{A}}} \sum_{i \in \mathcal{A}} (\hat{\theta}_i^{(s)} - \theta_i^{(s)})^2. \tag{4.1}$$

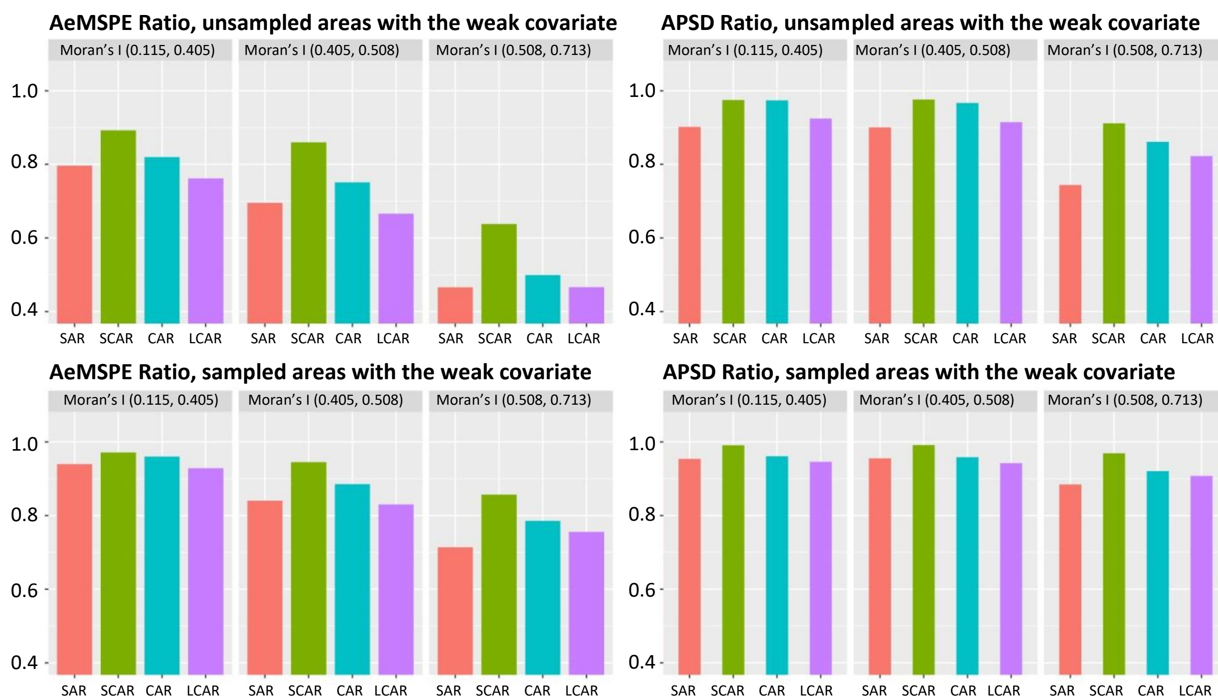
We also evaluate the uncertainty of the predictions using the average posterior standard deviation (APSD) defined as $S^{-1} \sum_{s=1}^{S} m_{\mathcal{A}}^{-1} \sum_{i \in \mathcal{A}} \text{sd}(\theta_i^{(s)}),$ where $\text{sd}(\theta_i^{(s)})$ is the posterior standard deviation of $\theta_i.$ By setting the independent FH model as a reference model, we consider the following ratios:

$$\text{AeMSPE} - \text{Ratio}_k = \frac{\text{AeMSPE}_k}{\text{AeMSPE}_1}, \quad \text{APSD} - \text{Ratio}_k = \frac{\text{APSD}_k}{\text{APSD}_1}, \tag{4.2}$$

where the subscript k indicates that the quantity is calculated from the posterior sample under the k^{th} model. These ratios measure improvements in AeMSPE and APSD achieved by fitting a spatial model over the independent FH model. A ratio less than 1 indicates superiority of the spatial model, otherwise the independent FH model is better. The smaller the ratio is, the more superior a spatial model is.

Model comparison: Various plots of Figure 4.1 summarize the ratios when the strong covariate $\mathbf{x}_1$ is excluded from the fitted models. We categorize AeMSPE-Ratios and APSD-Ratios under each replication into three groups based on the Moran's I values of θ_i's, namely the lowest, middle, and biggest thirds, respectively. The first row summarizes the prediction results for the seven non-sampled areas. As expected, spatial models show remarkable improvement in AeMSPE and APSD, and the improvements are greater when Moran's I values are larger. In terms of AeMSPE, the SAR and LCAR models produce at least 20%, 30%, and 50% more accurate predictions when Moran's I values are in the first, second, and third groups, respectively. In terms of the uncertainty of prediction, predictions of the SAR and LCAR models have 10% smaller APSD for the first and second groups. In the third group, the SAR model shows more than 25% reduction in APSD.

Figure 4.1 AeMSPE-Ratios and ASPDE-Ratios for predictions with the *weak* covariate.

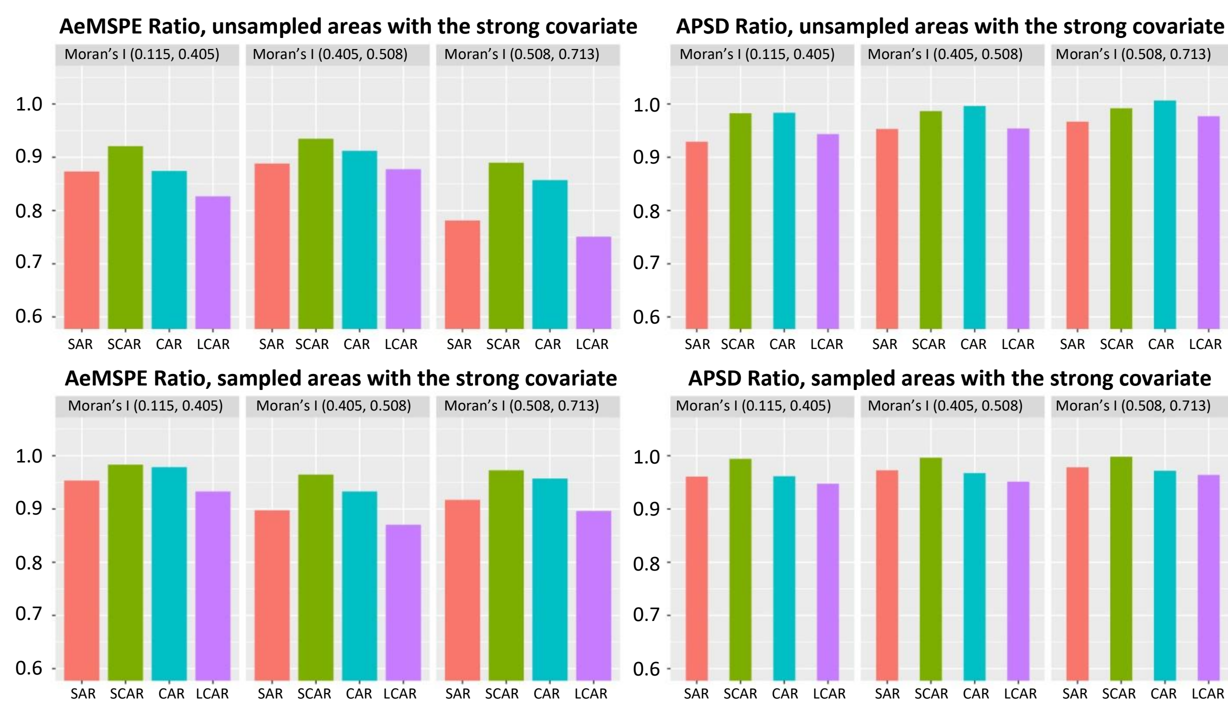


- Vertical bars are drawn for the ratios, where a bar shorter than 1 represents better (smaller) AeMSPE or APSD for the corresponding model relative to the independent FH model.
- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive, AeMSPE = Average empirical mean squared prediction error, APSD = Average posterior standard deviation.

For the sampled areas with direct estimates, the improvements in AeMSPE are less than 10% for the first group but more than 15% and 25% for the second and third groups (grouped by Moran's I values), respectively. Additionally, predictions from spatial models have a lower level of uncertainty, and for the third group, the SAR model predictions have more than 10% smaller ASPD.

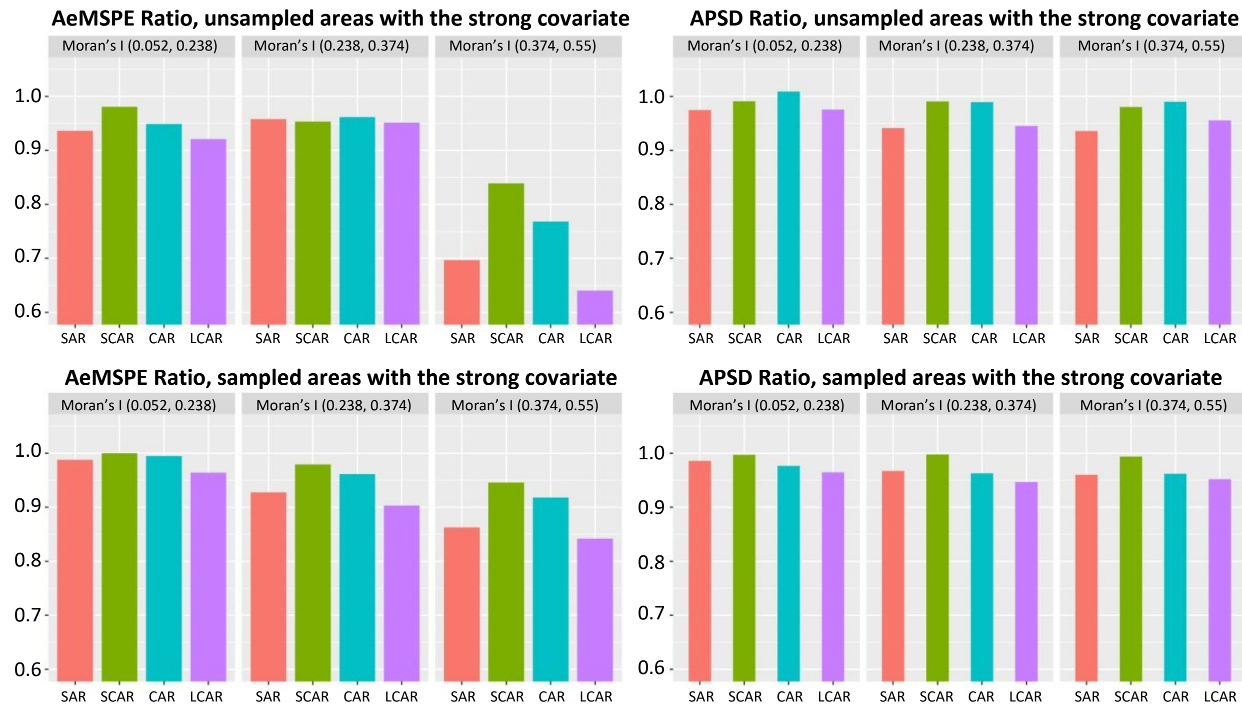
Similarly, various plots of Figure 4.2 summarize the ratios when the weak covariate $\mathbf{x}_2$ is excluded from the fitted models. In general, the spatial models continue to produce better predictions over the independent FH model. The LCAR model performs the best overall for the seven non-sampled areas, resulting in a reduction of over 10% in AeMSPE for the first and second groups, respectively, and around 25% in AeMSPE for the third group. In terms of uncertainty, the spatial models show more than 5% but less than 10% smaller APSD. For the sampled small areas, the SAR and LCAR models show an AeMSPE reduction of around 5%-13%, but APSD reductions are less than 5%. Unlike the previous results with the weak covariate, the improvements in AeMSPE and APSD are comparable across three groups that are categorized by Moran's I values. This is because the Moran's I values of small area means are mostly determined by the strong covariate, and once the strong covariate is present in the model to explain the spatial variability, the spatial variability in the residuals do not vary markedly across the three groups. We regroup the ratios in terms of the Moran's I values of the residuals obtained by regressing the strong covariate on the small area means, and summarize the ratios in Figure 4.3. Under this categorization, the LCAR model shows the best performance illustrating 5%, 10%, and 15% AeMSPE reductions for the first, second, and third groups, respectively. It can be also seen that the bigger the Moran's I value is, the more improvement the spatial models achieve.

Figure 4.2 AeMSPE-Ratios and ASPDE-Ratios predictions of non-sampled and sampled area means with the *strong* covariate.



- Vertical bars are drawn for the ratios, where a bar shorter than 1 represents better (smaller) AeMSPE or ASPDE for the corresponding model relative to the independent FH model.
- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive, AeMSPE = Average empirical mean squared prediction error, APSD = Average posterior standard deviation.

Figure 4.3 AeMSPE-Ratios and ASPDE-Ratios for predictions of non-sampled and sampled area means with the *strong* covariate. Results are grouped by the Moran's I values of the residuals, where the residuals are obtained by regressing the strong covariate on the small area means.



- Vertical bars are drawn for the ratios, where a bar shorter than 1 represents better (smaller) AeMSPE or ASPDE for the corresponding model relative to the independent FH model.
- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive, AeMSPE = Average empirical mean squared prediction error, APSD = Average posterior standard deviation.

5. Application to the Current Population Survey data

In this section, we evaluate the spatial models in terms of their prediction accuracy for some state-level population median incomes. The U.S. Department of Health and Human Service (HHS) annually needed accurate data for median incomes for states to implement a welfare program. While accurate national median income data are available from the Current Population Survey (CPS), the CPS data do not provide accurate state-level median income data. To supply accurate statistics to the HHS, the U.S. Census Bureau considered model-based small area estimation methods by utilizing auxiliary data from other federal programs. We apply proposed spatial models to estimate 1989 four-person family median incomes for the contiguous forty-nine U.S. states, including the District of Columbia. We use the direct estimates of 1990 CPS and compare our predictions with the more reliable statistics from the *long form* from the 1990 Census, i.e., we consider the statistics from the long form from the 1990 Census as the true values. Prediction performances are measured using all small areas and a subset of areas after leaving out multiple direct estimates.

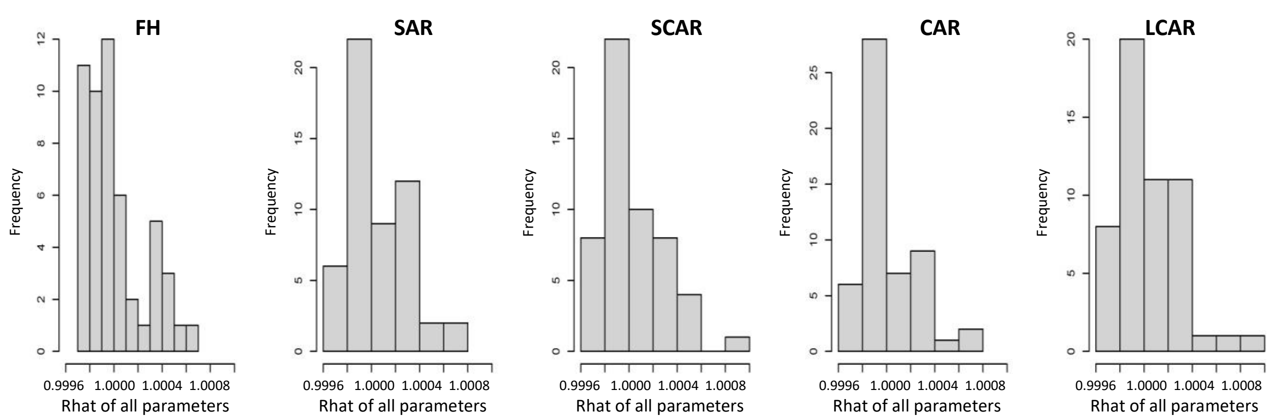
5.1 Four-person family median income estimation

Let θ_i be the true four-person family median income of the i^{th} state for the year 1989, where $i = 1, \ldots, 49.$ The states of Alaska and Hawaii are excluded as they are not geographically connected to the mainland. Let Y_i be the direct estimate of θ_i from the 1990 CPS. The covariates of interest are 1980 Census median income x_{i1} and an adjusted 1980 Census median income x_{i2}. The adjusted Census median income x_{i2} is defined as $\left(\text{PCI}_{i,1989} / \text{PCI}_{i,1979}\right) x_{i1},\ i = 1, \ldots, m,$ where $\text{PCI}_{i,1979}$ and $\text{PCI}_{i,1989}$ are the 1979 and 1989 per capita incomes of the i^{th} state provided by the Bureau of Economic Analysis of the U.S. Department of Commerce. It has been known that the adjusted Census median income is a good covariate which very effectively accounts for the variability of the small area median income.

With the noninformative prior (2.14) with $\alpha = 0,$ we fit all five models as described by (2.8)-(2.10) with $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_1, \mathbf{x}_2]$ and $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_1],$ where for the second covariate setting, we exclude the adjusted Census median income from the fitted model. For each model considered, we run 4 parallel HMC chains for 2,500 iterations after 5,000 burn-in iterations using `rstan` (Stan Development Team, 2018). We retain every 10^{th} iteration and concatenate the 4 chains to obtain a posterior sample of size 10,000. For all models and parameters, the potential scale reduction factors ($\hat{R};$ Gelman and Rubin, 1992) are all one indicating no lack of convergence. The potential scale reduction factors are provided in Figure 5.1 and Table 5.1.

Using the posterior means, $\hat{\theta}_i$'s, we calculate the squared prediction errors from respective θ_i's and obtain the mean squared prediction error (MSPE), defined in Section 4, by averaging the $m = 49$ squared deviations. The average posterior standard deviations (APSD) associated with $\hat{\theta}_i$'s are used to quantify the uncertainty of predictions, and the widely applicable information criterion (WAIC; Watanabe and Opper, 2010) is used to evaluate and compare the models, where a smaller WAIC value indicates a better model fit.

Figure 5.1 The potential scale reduction factor $\hat{R}$ of all parameters when no non-sampled area exists.



- All values are practically one indicating no evidence of lack of convergence.
- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

Table 5.1
The potential scale reduction factor $\hat{R}$ of the hyperparameter σ_v^2 and ρ and corresponding 95% upper confidence limit for the dataset with no non-sampled area

Hyperparameter	Covariate included	Potential scale reduction factor (upper 95% confidence limit)				
		FH	SAR	SCAR	CAR	LCAR
σ_v^2	x_1, x_2	0.995 (1.017)	0.987 (1.010)	0.995 (1.018)	0.985 (1.007)	1.008 (1.031)
	x_1	0.993 (1.015)	0.991 (1.012)	0.999 (1.022)	0.987 (1.009)	0.987 (1.01)
ρ	x_1, x_2	–	1.005 (1.028)	0.997 (1.023)	0.998 (1.036)	0.994 (1.017)
	x_1	–	1.005 (1.025)	0.997 (1.016)	0.998 (1.017)	1.019 (1.039)

Note: All upper limits are within the acceptable range, below 1.1 (Gelman, Carlin, Stern, Dunson, Vehtari and Rubin, 2013, Chapter 11.5), indicating no evidence of lack of convergence.
- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

Table 5.2 summarizes various evaluation measures we considered and the respective percentage improvements (PI) of MSPE, APSD. When both covariates are available, the LCAR model has approximately 14% smaller MSPE and 4% smaller APSD than the independent FH model. In terms of MSPE, the second best performing model is the SAR having approximately 9.5% smaller MSPE. When only x_1 (week covariate) is included in the fitted model, the SAR model has approximately 40% smaller MSPE and 14% smaller APSD than the independent FH model. The CAR and LCAR models show competitive performances having approximately 36% smaller MSPE and 13% smaller APSD over the independent FH model. By removing the strong covariate x_2 from the full model, the MSPE of the SAR and LCAR models increase approximately 66% and 84%, respectively, whereas the MSPE of the independent FH model increases more than 150%.

Table 5.2
Mean squared prediction error, average posterior standard deviation, and respective percentage improvements (PI) of spatial models over the independent FH model

Covariate Included	x_1 , x_2					x_1				
	MSPE	MSPE-PI	APSD	APSD-PI	WAIC	MSPE	MSPE-PI	APSD	APSD-PI	WAIC
FH	2.88	–	1.93	–	*259.06 (7.13)*	7.27	–	2.31	–	267.75 (8.44)
SAR	2.61	*9.55%*	1.94	0.34%	261.46 (7.47)	4.34	*40.22%*	1.98	*14.25%*	265.76 (8.16)
SCAR	3.03	-5.14%	1.95	-0.91%	259.37 (7.01)	5.62	22.62%	2.22	3.52%	263.41 (7.29)
CAR	2.64	8.47%	1.91	1.24%	261.61 (7.86)	4.62	*36.35%*	2.01	*12.97%*	*263.32 (7.96)*
LCAR	2.47	*14.50%*	1.85	*4.19%*	261.79 (8.01)	4.54	*37.51%*	1.97	*14.36%*	263.35 (8.08)

- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive, MSPE = Mean squared prediction error, APSD = Average posterior standard deviations, WAIC = Widely applicable information criterion.

In terms of the goodness of fit, the independent FH model shows the best fit (the smallest WAIC) when both covariates are included. Conversely, when only x_1 (week covariate) is included in the fitted model, the independent FH model shows the worst fit having the largest WAIC value. However, considering the standard errors given in parentheses, there is no significant difference in the model fit.

Table 5.3 summarizes the posterior distributions of ρ in terms of the posterior mean, mode, and standard deviation. When all covariates are included in the fitted model, the posterior distributions of ρ indicate no strong spatial dependency having posterior means centered around zero with large standard deviations. In contrast, when only x_1 (week covariate) is included in the fitted model, ρ becomes very significant illustrating the posterior distributions concentrated near the upper limit of its support.

In summary, when the weak covariate does not adequately explain existing spatial variation, the spatial models produce significantly better predictions accounting for the spatial variation. When no substantial spatial variation remains in the residual, they make marginally better predictions than the independent FH model without sacrificing model fit.

Table 5.3
Posterior mean/mode (standard deviation) of ρ

Covariate included	SAR	SCAR	CAR	LCAR
x_1, x_2	0.10/0.40 (0.48)	-0.06/0.04 (0.14)	0.21/0.83 (0.55)	0.57/0.80 (0.27)
x_1	0.76/0.80 (0.14)	0.14/0.17 (0.04)	0.93/0.99 (0.09)	0.85/0.97 (0.13)

Note: The first and second rows of the table respectively summarizes posterior distributions of ρ when both covariates and only x_1 are included in the fitted model.

- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

5.2 Estimation of some non-sampled state means excluding their CPS values

In this section, we evaluate the spatial models in terms of their prediction accuracy for non-sampled small areas using the 1980 Census median income x_1. Specifically, we randomly exclude CPS estimates (direct estimates) of multiple states at each instance and make predictions for θ_i's of the excluded states. As there are 49 small areas (states), we created 12 datasets that lack direct estimates for $m_1 = 4$ or 5 areas, where m_1 is the number of non-sampled small areas as in Section 2.2. Excluded states for each dataset are listed in Table 5.4.

For each dataset, we fit the independent FH model and four spatial models as specified in (2.11)-(2.13) with the noninformative prior (2.13) with $\alpha = 0$ running HMC chains under the same setting as in Section 5.1. The $\hat{R}$ values show no evidence of lack of convergence, where detailed values are provided in Figure 5.2 and Table 5.5. For each non-sampled area, the squared prediction error $\text{SPE}_i = (\hat{\theta}_i - \theta_i)^2$ and posterior standard deviation $\text{sd}(\theta_i)$ are obtained for each model. Based on these, the prediction performance is compared with the following ratio: for $i = 1, \ldots, m$ and $k = 2, \ldots, 5,$

$$\text{SPE} - \text{Ratio}_{ki} = \frac{\text{SPE}_{ki}}{\text{SPE}_{1i}}, \quad \text{PSD} - \text{Ratio}_{ki} = \frac{\text{sd}_k(\theta_i)}{\text{sd}_1(\theta_i)} \tag{5.1}$$

where $\text{sd}_k(\theta_i)$ is the posterior standard deviation of θ_i under the k^{th} model. A value of $\text{SPE} - \text{Ratio}_{ki}$ $(\text{PSD} - \text{Ratio}_{ki})$ less than one indicates that the k^{th} spatial model has a smaller squared prediction error (posterior standard deviation) than the independent FH model. In Figures 5.3 and 5.4, we display the ratios using a red and blue color scheme to denote ratios greater than and less than one, respectively, where a darker color represents a more extreme value.

Overall, SAR, SCAR, CAR, and LCAR models exhibit smaller SPEs in 35, 41, 36, and 36 states, respectively. The SCAR model has the greatest number of states in which its predictions perform better than those of the independent FH model, but the overall improvements are the least. In more than 35 states, the SAR, CAR, and LCAR models produce more accurate predictions than the independent FH model, and in three states (New Mexico, Oregon, and Wisconsin), their SPEs are more than 100 times smaller. For California, Minnesota, and South Carolina, all spatial models make worse predictions than the independent FH model. California and Minnesota have much higher median incomes than surrounding states, whilst South Carolina has a significantly lower median income. Among the 49 states, these three states have the second, seventh, and nineteenth smallest local Moran's I values. This illustrates that if the small area mean of an non-sampled area is significantly different from the means of surrounding areas, then the spatial models may produce inferior predictions. The model that demonstrates the best fit in terms of WAIC is either the SCAR, CAR, or LCAR model, where the exact numbers are provided in Table 5.4.

Table 5.4
States whose CPS estimates are excluded for each dataset and corresponding widely applicable information criterion (WAIC)

Excluded states	FH	SAR	SCAR	CAR	LCAR
AZ MS OK SD	246.15 (7.79)	244.37 (7.50)	*241.71 (6.67)*	242.46 (7.50)	242.21 (7.57)
AR CO DE TN	245.79 (7.83)	241.23 (7.69)	241.05 (6.70)	*239.70 (7.61)*	239.85 (7.69)
MD MI NV WV	246.06 (7.83)	242.23 (7.58)	241.34 (6.78)	240.13 (7.32)	*240.02 (7.43)*
MT NC NE NY	248.91 (8.10)	245.97 (9.51)	243.85 (6.87)	242.88 (8.34)	*242.40 (8.36)*
DC GA ID ND	245.07 (7.91)	241.02 (6.61)	240.09 (6.44)	*239.16 (6.75)*	239.48 (6.75)
AL MO VT WY	245.57 (8.31)	245.13 (7.67)	*242.74 (7.38)*	242.89 (7.65)	243.36 (7.66)
FL LA UT WA	247.38 (7.39)	245.64 (7.29)	*243.08 (6.44)*	243.25 (7.20)	243.48 (7.33)
MA MN SC TX	248.32 (9.73)	242.43 (8.95)	244.75 (8.83)	240.99 (8.69)	*240.34 (8.56)*
KY RI VA WI	243.86 (8.09)	241.74 (7.06)	240.05 (6.76)	239.04 (6.91)	*238.87 (6.90)*
IL IN NH PA	244.69 (7.10)	245.12 (7.41)	*243.31 (6.59)*	244.45 (7.60)	244.39 (7.68)
CA ME NJ OH	248.62 (8.54)	244.06 (8.26)	245.28 (7.68)	*242.00 (7.78)*	242.01 (7.90)
CT IA KS NM OR	239.86 (7.95)	239.81 (7.76)	*237.28 (7.29)*	237.78 (7.75)	237.75 (7.60)

Note: The numbers in parentheses are the standard errors of WAIC estimates.
- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.
- AL = Alabama, AZ = Arizona, AR = Arkansas, CA = California, CO = Colorado, CT = Connecticut, DE = Delaware, DC = District of Columbia, FL = Florida, GA = Georgia, ID = Idaho, IL = Illinois, IN = Indiana, IA = Iowa, KS = Kansas, KY = Kentucky, LA = Louisiana, ME = Maine, MD = Maryland, MA = Massachusetts, MI = Michigan, MN = Minnesota, MS = Mississippi, MO = Missouri, MT = Montana, NE = Nebraska, NV = Nevada, NH = New Hampshire, NJ = New Jersey, NM = New Mexico, NY = New York, NC = North Carolina, ND = North Dakota, OH = Ohio, OK = Oklahoma, OR = Oregon, PA = Pennsylvania, RI = Rhode Island, SC = South Carolina, SD = South Dakota, TN = Tennessee, TX = Texas, UT = Utah, VT = Vermont, VA = Virginia, WA = Washington, WV = West Virginia, WI = Wisconsin, WY = Wyoming.

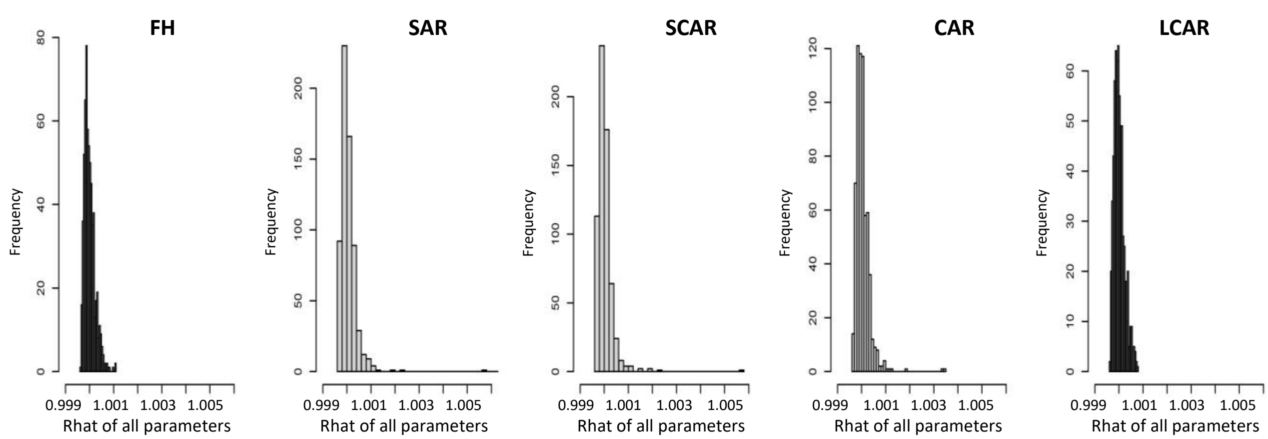
Table 5.5
The potential scale reduction factor $\hat{R}$ of the hyperparameter σ_v^2 and ρ and corresponding 95% upper confidence limit for the 12 datasets with non-sampled areas

	Excluded states	FH	SAR	SCAR	CAR	LCAR
Potential scale	AZ MS OK SD	0.991 (1.013)	1.012 (1.036)	0.995 (1.017)	1.005 (1.028)	1.005 (1.030)
reduction factor	AR CO DE TN	0.997 (1.019)	1.010 (1.034)	0.999 (1.023)	0.985 (1.008)	0.984 (1.007)
(upper 95%	MD MI NV WV	0.992 (1.015)	1.007 (1.030)	1.023 (1.047)	1.001 (1.025)	1.011 (1.035)
confidence limit)	MT NC NE NY	0.990 (1.012)	0.984 (1.009)	0.997 (1.020)	0.999 (1.024)	0.983 (1.006)
of σ_v^2	DC GA ID ND	1.002 (1.024)	1.007 (1.032)	1.001 (1.024)	1.011 (1.036)	0.997 (1.020)
	AL MO VT WY	0.997 (1.019)	1.006 (1.030)	0.998 (1.021)	1.002 (1.026)	1.019 (1.045)
	FL LA UT WA	1.014 (1.037)	1.005 (1.027)	1.002 (1.025)	0.999 (1.023)	1.019 (1.043)
	MA MN SC TX	1.008 (1.031)	1.001 (1.025)	1.003 (1.025)	0.994 (1.019)	1.007 (1.032)
	KY RI VA WI	1.011 (1.034)	1.005 (1.029)	1.009 (1.033)	0.989 (1.011)	1.011 (1.035)
	IL IN NH PA	0.992 (1.013)	1.018 (1.041)	0.992 (1.013)	0.989 (1.014)	0.989 (1.012)
	CA ME NJ OH	1.002 (1.025)	1.009 (1.033)	1.001 (1.024)	1.011 (1.035)	1.003 (1.026)
	CT IA KS NM OR	0.997 (1.018)	1.008 (1.032)	1.007 (1.030)	1.009 (1.034)	1.012 (1.041)
Potential scale	AZ MS OK SD	–	1.005 (1.026)	1.001 (1.029)	1.004 (1.040)	0.995 (1.018)
reduction factor	AR CO DE TN	–	1.019 (1.042)	0.993 (1.018)	1.000 (1.034)	1.005 (1.028)
(upper 95%	MD MI NV WV	–	1.005 (1.027)	0.992 (1.017)	0.988 (1.021)	1.002 (1.026)
confidence limit)	MT NC NE NY	–	1.001 (1.024)	0.999 (1.030)	0.986 (1.023)	0.998 (1.023)
of ρ	DC GA ID ND	–	0.999 (1.020)	0.995 (1.024)	0.991 (1.026)	1.000 (1.024)
	AL MO VT WY	–	1.003 (1.026)	1.011 (1.037)	0.989 (1.025)	1.008 (1.031)
	FL LA UT WA	–	1.004 (1.025)	0.996 (1.026)	1.000 (1.033)	1.005 (1.029)
	MA MN SC TX	–	1.002 (1.041)	1.014 (1.045)	0.984 (1.018)	0.996 (1.02)
	KY RI VA WI	–	1.010 (1.032)	0.990 (1.016)	0.993 (1.031)	0.986 (1.009)
	IL IN NH PA	–	1.030 (1.055)	0.993 (1.016)	0.994 (1.029)	1.003 (1.025)
	CA ME NJ OH	–	1.003 (1.026)	0.997 (1.026)	0.996 (1.032)	0.989 (1.012)
	CT IA KS NM OR	–	1.017 (1.040)	0.997 (1.022)	1.009 (1.044)	0.981 (1.003)

Note: All upper limits are within the acceptable range, below 1.1 (Gelman et al., 2013, Chapter 11.5), indicating no evidence of lack of convergence.

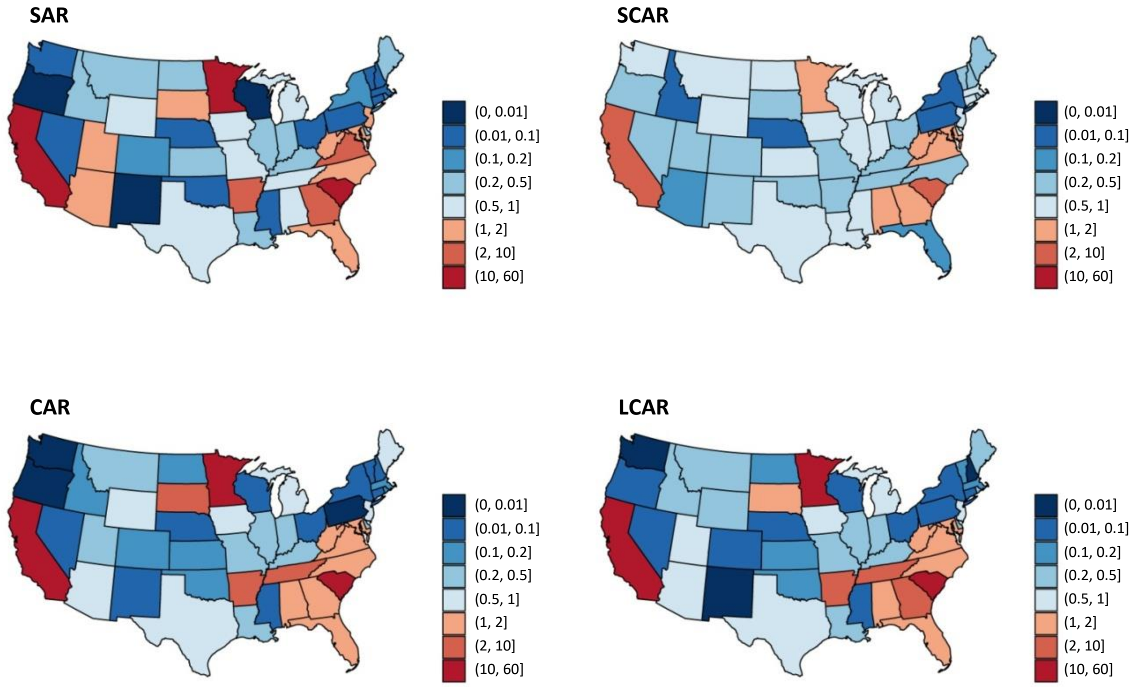
- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.
- AL = Alabama, AZ = Arizona, AR = Arkansas, CA = California, CO = Colorado, CT = Connecticut, DE = Delaware, DC = District of Columbia, FL = Florida, GA = Georgia, ID = Idaho, IL = Illinois, IN = Indiana, IA = Iowa, KS = Kansas, KY = Kentucky, LA = Louisiana, ME = Maine, MD = Maryland, MA = Massachusetts, MI = Michigan, MN = Minnesota, MS = Mississippi, MO = Missouri, MT = Montana, NE = Nebraska, NV = Nevada, NH = New Hampshire, NJ = New Jersey, NM = New Mexico, NY = New York, NC = North Carolina, ND = North Dakota, OH = Ohio, OK = Oklahoma, OR = Oregon, PA = Pennsylvania, RI = Rhode Island, SC = South Carolina, SD = South Dakota, TN = Tennessee, TX = Texas, UT = Utah, VT = Vermont, VA = Virginia, WA = Washington, WV = West Virginia, WI = Wisconsin, WY = Wyoming.

Figure 5.2 The potential scale reduction factor $\hat{R}$ of all parameters, where $\hat{R}$ values of the 12 datasets are all combined.



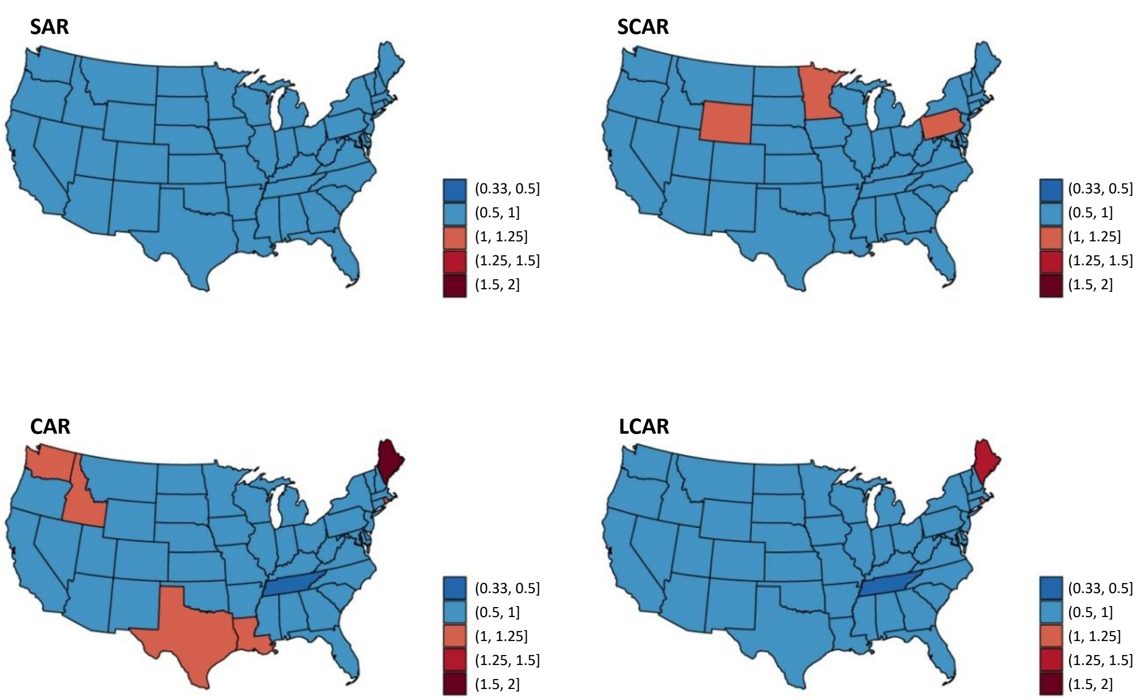
- All values are practically one indicating no evidence of lack of convergence.
- FH = Fay-Herriot, SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

Figure 5.3 SPE-Ratios.



- The values smaller (larger) than one are represented in blue (red) color and indicate a spatial model has a smaller (larger) squared deviation.
- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

Figure 5.4 PSD-Ratios.



- The values smaller (larger) than one are represented in blue (red) color and indicate a spatial model has a smaller (larger) posterior standard deviation.
- SAR = Simultaneous autoregressive, SCAR = Simple conditional autoregressive, CAR = Conditional autoregressive, LCAR = Leroux's conditional autoregressive.

6. Conclusions

In this paper, we followed a Bayesian approach to investigate four spatial random-effects models as alternatives to the independent Fay-Herriot model to estimate small area means. In particular, we considered four spatial models with different autocorrelation structures. We further extended the spatial models to allow multiple small areas without any direct estimates in predicting small area means for all the areas. For a class of noninformative priors, we established the propriety of posterior densities of the proposed models for both setups.

A simulation study in Section 4 illustrates that prediction accuracy can be greatly improved by considering spatial models when effective covariates are unavailable. Datta, Hall and Mandal (2011) noted that the prediction accuracy of small area estimation models largely depends on the availability of good covariates. In other words, when suitable covariates are unavailable, the independent Fay-Herriot model may not provide a significant advantage over direct estimates. The simulation results indicated that, in such cases, the spatial models considerably increase the prediction accuracy by exploiting information from adjacent areas.

We applied the proposed spatial random-effects models to estimate four-person family median incomes. Even when a good covariate exists, the spatial models exhibited noticeable improvements in terms of mean squared deviation and average posterior standard deviations. When a good covariate is unavailable, spatial models provided significantly more accurate median income predictions with much smaller variability, which agrees with the simulation results. Furthermore, the SAR and LCAR models provide more precise small area estimates when direct estimates of some states are excluded in model fitting.

In summary, the spatial models considered in this paper outperform the independent Fay-Herriot model. A significant improvement can be expected when effective covariates are unavailable. Since useful covariates are not always available, the utility of the proposed models in small area estimation can be substantial. Our simulation study and real data analysis demonstrate no clear winner among the proposed models. Nonetheless, the SAR and LCAR models show better performance compared with other spatial models. Also, the LCAR model performs robustly well with simulated data from the SAR model and real data with unknown spatial dependence. Thus, in the context of real applications where true dependency is unknown, we recommend the LCAR model.

This work assumes that all areas have at least one neighborhood. In real applications, however, there are many situations that data contain small areas with no neighborhood (stand-alone areas). Although the proposed models can accommodate stand-alone areas by adjusting the diagonal entries of the precision matrices as in Brown, Datta and Lazar (2017), we find that this approach results in a counterintuitive prior, where stand-alone areas have smaller prior random effect variances than areas with neighborhoods. Also, we find that this prior can considerably deteriorate predictions of stand-alone areas. This is a practically important problem as many countries have islands, and this will possibly be our future research to pursue.

Disclaimer and acknowledgements

This report is released to inform interested parties of ongoing research and to encourage discussion. The views expressed on statistical, methodological, technical, or operational issues are those of the authors and not those of the U.S. Census Bureau, or the University of Georgia. The authors are grateful to Dr. William R. Bell for his insightful comments on an earlier version of this work that led to an improved manuscript.

Appendix

A. Proof of the propriety of the posterior pdf

Proof of Theorem 1. For convenience of notation, we denote $\mathbf{\Omega}_k(\rho)$ by $\mathbf{\Omega}_k$, and, for a given square matrix $\mathbf{A}$, the determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$. We use K to denote a generic positive constant, not depending on the variables we are integrating out.

Let $m_1 \geq 0$ be the number of small areas with no direct estimates and let $m_2 = m - m_1$. Also, let $\mathbf{Y}_{(2)}$ be the $m_2 \times 1$ vector with direct estimates of the sampled small areas. Without loss of generality, we assume that $\theta_1, \ldots, \theta_m$ are arranged so that $\boldsymbol{\theta} = (\boldsymbol{\theta}_{(1)}^\top, \boldsymbol{\theta}_{(2)}^\top)^\top$. Let $\mathbf{D}_{(2)} = \{D_i\}_{i=m_1+1}^m$ be the diagonal matrix with sampling variances corresponding to the components of $\mathbf{Y}_{(2)}$ and $\delta = \max_{m_1 < i \leq m} D_i < \infty$.

The joint pdf of $\mathbf{Y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2$ and ρ is given by

$$f(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho) = N_{m_2}(\mathbf{y}_{(2)} \,|\, \boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)})\, N_m(\boldsymbol{\theta} \,|\, \mathbf{X}\boldsymbol{\beta}, \sigma_v^2 \mathbf{\Omega}^{-1})\, g(\sigma_v^2)\, h(\rho), \tag{A.1}$$

where $N_{m_2}(\mathbf{y}_{(2)} \,|\, \boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)})$ is the normal pdf with mean $\boldsymbol{\theta}_{(2)}$ and covariance matrix $\mathbf{D}_{(2)}$. The posterior pdf $\pi(\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho \,|\, \mathbf{y}_{(2)})$ will be proper if and only if the function $f(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho)$ is integrable with respect to $\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2$ and ρ. Since

$$N_{m_2}(\mathbf{y}_{(2)} \,|\, \boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)}) \leq K \exp\left\{-\frac{1}{2\delta}(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})^\top (\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})\right\},$$

we have from (A.1)

$$\begin{aligned} f(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho) &\leq K \exp\left\{-\frac{1}{2\delta}(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})^\top (\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})\right\} N_m(\boldsymbol{\theta} \,|\, \mathbf{X}\boldsymbol{\beta}, \sigma_v^2 \mathbf{\Omega}^{-1})\, g(\sigma_v^2)\, h(\rho) \\ &= K \int \exp\left\{-\frac{1}{2\delta}(\mathbf{y} - \boldsymbol{\theta})^\top (\mathbf{y} - \boldsymbol{\theta})\right\} d\mathbf{y}_{(1)}\, N_m(\boldsymbol{\theta} \,|\, \mathbf{X}\boldsymbol{\beta}, \sigma_v^2 \mathbf{\Omega}^{-1})\, g(\sigma_v^2)\, h(\rho), \end{aligned} \tag{A.2}$$

where $\mathbf{y} = (\mathbf{y}_{(1)}^\top, \mathbf{y}_{(2)}^\top)^\top$. By integrating both sides of (A.2) with respect to $\boldsymbol{\theta}$, we get

$$\int f(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho)\, d\boldsymbol{\theta} \leq K g(\sigma_v^2)\, h(\rho) \int N_m(\mathbf{y} \,|\, \mathbf{X}\boldsymbol{\beta}, \delta \mathbf{I}_m + \sigma_v^2 \mathbf{\Omega}^{-1})\, d\mathbf{y}_{(1)}. \tag{A.3}$$

Partition $\mathbf{X}$ as $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]^\top$, where $\mathbf{X}_1^\top$ is $m_1 \times p$ and $\mathbf{X}_2^\top$ is $m_2 \times p$. We assume that $\operatorname{rank}(\mathbf{X}_2) = p$. Let $\mathbf{d} = (\mathbf{0}_{m_1}^\top, \mathbf{y}_{(2)}^\top)^\top$, $\boldsymbol{\phi} = (\mathbf{y}_{(1)}^\top, \boldsymbol{\beta}^\top)^\top$ and

$$\mathbf{G} = \begin{bmatrix} -\mathbf{I}_{m_1} & \mathbf{X}_1^\top \\ \mathbf{0}_{m_2,\, m_1} & \mathbf{X}_2^\top \end{bmatrix}.$$

Then, we can write

$$\mathbf{y} - \mathbf{X}\boldsymbol{\beta} = \mathbf{d} - \mathbf{G}\boldsymbol{\phi},$$

where $\mathbf{G}$ is $m \times (m_1 + p)$, $\boldsymbol{\phi}$ is $(m_1 + p) \times 1$. Hence, (A.3) can be written as

$$\int f\,(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho)\, d\,\boldsymbol{\theta} \;\le\; Kg\,(\sigma_v^2)\, h(\rho) \int N_m\,(d \mid \mathbf{G}\boldsymbol{\phi}, \delta\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}^{-1})\, d\mathbf{y}_{(1)}. \tag{A.4}$$

By integrating both sides of (A.4) with respect to $\boldsymbol{\beta}$, we get

$$\int f\,(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho)\, d\,\boldsymbol{\theta}\, d\,\boldsymbol{\beta} \;\le\; Kg\,(\sigma_v^2)\, h(\rho) \int N_m\,(\mathbf{d} \mid \mathbf{G}\boldsymbol{\phi}, \delta\mathbf{I}_m + \sigma_v^2\boldsymbol{\Omega}^{-1})\, d\boldsymbol{\phi}. \tag{A.5}$$

Since $\operatorname{rank}(\mathbf{X}_2) = p$, we immediately get that $\operatorname{rank}(\mathbf{G}) = m_1 + p$. Thus $\mathbf{G}$ has full column rank. We denote $m_1 + p$ by q. For $k = 2, \ldots, 5$, we now derive upper bounds for

$$\left|\, \delta\,\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}_k^{-1} \right|^{-1/2} \int \exp\left\{ -\frac{1}{2}\,(\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top\, (\delta\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}_k^{-1})^{-1}\, (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) \right\} d\boldsymbol{\phi}$$

which will be integrable with respect to σ_v^2 and ρ as follows.

A.1 Details for the SCAR Model

We first consider the CAR model where $k = 2$. Let $\mathbf{P}_\mathbf{W}$ be an orthogonal matrix such that $\mathbf{P}_\mathbf{W}^\top\, \mathbf{W}\mathbf{P}_\mathbf{W} = \operatorname{diag}\{\lambda_i\}_{i=1}^m = \boldsymbol{\Lambda}.$ Then $\boldsymbol{\Omega}_2\,(\rho)^{-1} = \mathbf{P}_\mathbf{W}\, \{\mathbf{I} - \rho\boldsymbol{\Lambda}\}^{-1}\, \mathbf{P}_\mathbf{W}^\top,$ and hence,

$$\begin{aligned}(\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top\, (\delta\,\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}_2^{-1})^{-1}\, (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) &= (\mathbf{P}_\mathbf{W}^\top\, \mathbf{d} - \mathbf{P}_\mathbf{W}^\top\, \mathbf{G}\boldsymbol{\phi})^\top \left(\delta\,\mathbf{I}_m + \sigma_v^2\, \{\mathbf{I} - \rho\boldsymbol{\Lambda}\}^{-1}\right)^{-1} (\mathbf{P}_\mathbf{W}^\top\, \mathbf{d} - \mathbf{P}_\mathbf{W}^\top\, \mathbf{G}\boldsymbol{\phi}) \\ &= (\mathbf{d}_* - \mathbf{G}_*\, \boldsymbol{\phi})^\top \left(\delta\,\mathbf{I}_m + \sigma_v^2\, \{\mathbf{I} - \rho\,\boldsymbol{\Lambda}\}^{-1}\right)^{-1} (\mathbf{d}_* - \mathbf{G}_*\, \boldsymbol{\phi}),\end{aligned}$$

where $\mathbf{d}_* = \mathbf{P}_\mathbf{W}^\top\, \mathbf{d}$ and $\mathbf{G}_* = \mathbf{P}_\mathbf{W}^\top\, \mathbf{G}.$ Suppose the rows of $\mathbf{G}_*$ corresponding to distinct q indices $\{i_1, \ldots, i_q\} \subseteq \{1, \ldots, m\}$ are linearly independent. We denote these rows by $\mathbf{g}_{i_k^*}^\top,$ $k = 1, \ldots, q.$ Let $\mathbf{A}$ be the $q \times q$ non-singular matrix $[\mathbf{g}_{i_1^*}, \ldots, \mathbf{g}_{i_q^*}]^\top$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_q)^\top = \mathbf{A}\boldsymbol{\phi}.$ Note that

$$(\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top\, (\delta\,\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}_2^{-1})^{-1}\, (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) \;\ge\; \sum_{k=1}^{q} \frac{(d_{i_k^*} - \eta_{i_k})^2}{\delta + \sigma_v^2\, (1 - \rho\,\lambda_{i_k})^{-1}}.$$

From this, we get that

$$\begin{aligned}\int \exp\left\{ -\frac{1}{2}\,(\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top (\delta\,\mathbf{I}_m + \sigma_v^2\, \boldsymbol{\Omega}_2^{-1})^{-1} (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) \right\} d\boldsymbol{\phi} &\le \int \exp\left\{ -\frac{1}{2} \sum_{k=1}^{q} \frac{(d_{i_k^*} - \eta_{i_k})^2}{\delta + \sigma_v^2\, (1 - \rho\,\lambda_{i_k})^{-1}} \right\} d\boldsymbol{\phi} \\ &= \int \exp\left\{ -\frac{1}{2} \sum_{k=1}^{q} \frac{(d_{i_k^*} - \eta_{i_k})^2}{\delta + \sigma_v^2 (1 - \rho\lambda_{i_k})^{-1}} \right\} d\boldsymbol{\eta} \left| \mathbf{A}^\top \mathbf{A} \right|^{-1/2} \\ &= K \prod_{k=1}^{q} \left\{ \delta + \sigma_v^2\, (1 - \rho\,\lambda_{i_k})^{-1} \right\}^{1/2},\end{aligned} \tag{A.6}$$

where $K > 0$ is a finite generic constant. Also, we know that

$$\left| \delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_2^{-1} \right|^{-1/2} = \prod_{i=1}^{m} \left\{ \delta + \sigma_v^2 \, (1 - \rho \lambda_i)^{-1} \right\}^{-1/2}. \tag{A.7}$$

By (A.6) and (A.7), we get

$$\begin{aligned}
\left| \delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_2^{-1} \right|^{-1/2} \int \exp\left\{ -\frac{1}{2} (\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top (\delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_2^{-1})^{-1} (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) \right\} d\boldsymbol{\phi} \\
\le K \prod_{i \notin \{i_1, \ldots, i_q\}} \left\{ \delta + \sigma_v^2 \, (1 - \rho \lambda_i)^{-1} \right\}^{-1/2} \\
\le K \left\{ I\,(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-q)/2} \prod_{i \notin \{i_1, \ldots, i_q\}} (1 - \rho \, \lambda_i)^{1/2} \, I\,(\sigma_v^2 > N) \right\}
\end{aligned} \tag{A.8}$$

for any positive number N. Recall that $\lambda_m^{-1} < \rho < \lambda_1^{-1}$. We know $1 - \rho \, \lambda_i$ is an eigenvalue of $\boldsymbol{\Omega}_2$. Thus, for $\lambda_m^{-1} < \rho < \lambda_1^{-1}$, for $i = 1, \ldots, m$, $1 - \rho \lambda_i > 0$. Also, $\sum_{i=1}^{m} (1 - \rho \lambda_i) = m$. These imply that $0 < 1 - \rho \lambda_i < m$. Then from (A.8), we get

$$\begin{aligned}
\left| \delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_2^{-1} \right|^{-1/2} \int \exp\left\{ -\frac{1}{2} (\mathbf{d} - \mathbf{G}\boldsymbol{\phi})^\top (\delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_2^{-1})^{-1} (\mathbf{d} - \mathbf{G}\boldsymbol{\phi}) \right\} d\boldsymbol{\phi} \\
\le K \left\{ I\,(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-q)/2} \, I\,(\sigma_v^2 > N) \right\}.
\end{aligned} \tag{A.9}$$

From (A.5) and (A.9), it follows under the conditions of the theorem that the desired integral $\int f\,(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho) \, d\boldsymbol{\theta} d\boldsymbol{\beta} d\sigma_v^2 d\rho$ is finite.

A.2 Details for the SAR Model

We now consider $k = 3$ for the SAR model. With $\mathbf{W}_* = \mathbf{L}^{-1/2} \mathbf{W} \mathbf{L}^{-1/2}$, we have

$$\begin{aligned}
\boldsymbol{\Omega}_3 &= (\mathbf{I}_m - \rho \tilde{\mathbf{W}})^\top (\mathbf{I}_m - \rho \tilde{\mathbf{W}}) \\
&= (\mathbf{L} - \rho \mathbf{W})^\top \mathbf{L}^{-2} (\mathbf{L} - \rho \mathbf{W}) \\
&= \mathbf{L}^{1/2} (\mathbf{I}_m - \rho \mathbf{W}_*) \, \mathbf{L}^{-1} (\mathbf{I}_m - \rho \mathbf{W}_*) \mathbf{L}^{1/2}.
\end{aligned}$$

First, $\operatorname{tr} \boldsymbol{\Omega}_3 = m + \rho^2 \sum_i \sum_j \tilde{w}_{ij}^2 \le m + \rho^2 \sum_i \sum_j \tilde{w}_{ij} = m + \rho^2 m < 2m$ since $0 \le \tilde{w}_{ij} \le 1$, $\sum_j \tilde{w}_{ij} = 1$, and $-1 < \rho < 1$.

Note that the eigenvalues $\nu_1, \ldots, \nu_m$ of $\mathbf{W}_*$ are all real (since $\mathbf{W}_*$ is symmetric). Also, $\mathbf{W}_*$ and $\tilde{\mathbf{W}}$ have identical eigenvalues. Being a stochastic matrix, $\tilde{\mathbf{W}}$ has at least one eigenvalue which is one and the remaining eigenvalues are bounded above by 1, that is $|\nu_i| \le 1$ and $\max_i \nu_i = 1$. As $-1 < \rho < 1$ and $1 - \rho \nu_i > 0$, $|\boldsymbol{\Omega}_3| = \prod_{i=1}^{m} (1 - \rho \nu_i)^2 > 0$. Thus, the eigenvalues of $\boldsymbol{\Omega}_3$ are positive and bounded above by $2m$. Let $l_{(1)} = \min l_i$ and $l_{(m)} = \max l_i$, where $\mathbf{L} = \operatorname{diag}\{l_i\}_{i=1}^{m}$. Then $l_{(1)} > 0$ and $l_{(m)}$ is bounded above. By writing

$$\boldsymbol{\Sigma}_3 = \delta \mathbf{I}_m + \sigma_v^2 \, \boldsymbol{\Omega}_3^{-1} = \mathbf{L}^{-1/2} \left\{ \delta \mathbf{L} + \sigma_v^2 \, (\mathbf{I}_m - \rho \, \mathbf{W}_*)^{-1} \, \mathbf{L} \, (\mathbf{I} - \rho \, \mathbf{W}_*)^{-1} \right\} \mathbf{L}^{-1/2},$$

we have

$$\left|\boldsymbol{\Sigma}_3\right| = \left|\mathbf{L}\right|^{-1}\left|\delta\mathbf{L}+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\mathbf{L}(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\right| \geq \left|\mathbf{L}\right|^{-1} l_{(1)}^m \left|\delta\mathbf{I}_m+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-2}\right|$$
$$= \left|\mathbf{L}\right|^{-1} l_{(1)}^m \prod_{i=1}^m \left\{\delta+\sigma_v^2(1-\rho\nu_i)^{-2}\right\}, \quad \text{(A.10)}$$

Letting $\mathbf{P}_{\mathbf{W}_*}$ be the matrix of eigenvectors of $\mathbf{W}_*$ such that $\mathbf{P}_{\mathbf{W}_*}^\top\mathbf{W}_*\mathbf{P}_{\mathbf{W}_*}=\operatorname{diag}\{\nu_i\}_{i=1}^m=\mathbf{N}_*$, we also have

$$\begin{aligned}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})^\top\boldsymbol{\Sigma}_3^{-1}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi}) &= (\mathbf{L}^{1/2}\mathbf{d}-\mathbf{L}^{1/2}\mathbf{G}\boldsymbol{\phi})^\top\left\{\delta\mathbf{L}+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\mathbf{L}(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\right\}^{-1}(\mathbf{L}^{1/2}\mathbf{d}-\mathbf{L}^{1/2}\mathbf{G}\boldsymbol{\phi})\\
&= (\mathbf{r}-\mathbf{S}\boldsymbol{\phi})^\top\left\{\delta\mathbf{L}+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\mathbf{L}(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\right\}^{-1}(\mathbf{r}-\mathbf{S}\boldsymbol{\phi})\\
&\geq (l_{(m)}^{-1/2}\mathbf{r}-l_{(m)}^{-1/2}\mathbf{S}\boldsymbol{\phi})^\top\left\{\delta\mathbf{I}_m+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-2}\right\}^{-1}(l_{(m)}^{-1/2}\mathbf{r}-l_{(m)}^{-1/2}\mathbf{S}\boldsymbol{\phi})\\
&= (\tilde{\mathbf{r}}-\tilde{\mathbf{S}}\boldsymbol{\phi})^\top\left\{\delta\mathbf{I}_m+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{N}_*)^{-2}\right\}^{-1}(\tilde{\mathbf{r}}-\tilde{\mathbf{S}}\boldsymbol{\phi})\\
&\geq \sum_{k=1}^q \frac{(\tilde{r}_{i_k}-\tilde{s}_{i_k}^\top\boldsymbol{\phi})^2}{\delta+\sigma_v^2(1-\rho\nu_{i_k})^{-2}}, \quad \text{(A.11)}\end{aligned}$$

where $\mathbf{r}=\mathbf{L}^{1/2}\mathbf{d}$, $\mathbf{S}=\mathbf{L}^{1/2}\mathbf{G}$, $\tilde{\mathbf{r}}=l_{(m)}^{-1/2}\mathbf{P}_{\mathbf{W}_*}\mathbf{r}$, $\tilde{\mathbf{S}}=l_{(m)}^{-1/2}\mathbf{P}_{\mathbf{W}_*}\mathbf{S}$, and $\{i_1,\ldots,i_q\}$ is a subset of $\{1,\ldots,m\}$ so that the $q\times q$ matrix $[\tilde{\mathbf{s}}_{i_1},\ldots,\tilde{\mathbf{s}}_{i_q}]^\top=\tilde{\mathbf{S}}_1$, a submatrix of $\tilde{\mathbf{S}}$, is non-singular. Note that $\tilde{\mathbf{S}}_1$ is determined by $\mathbf{W}$. Using (A.11) we get

$$\int\exp\left\{-\frac{1}{2}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})^\top\boldsymbol{\Sigma}_3^{-1}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})\right\}d\boldsymbol{\phi}\leq(2\pi)^{q/2}\left|\tilde{\mathbf{S}}_1^\top\tilde{\mathbf{S}}_1\right|^{-1/2}\prod_{k=1}^q\left\{\delta+\sigma_v^2(1-\rho\nu_{i_k})^{-2}\right\}^{1/2}. \quad \text{(A.12)}$$

Based on (A.10) and (A.12), we get that

$$\begin{aligned}\int\left|\boldsymbol{\Sigma}_3\right|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})^\top\boldsymbol{\Sigma}_3^{-1}(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})\right\}d\boldsymbol{\phi} &\leq K\prod_{i\notin\{i_1,\ldots,i_q\}}\left\{\delta+\sigma_v^2(1-\rho\nu_i)^{-2}\right\}^{-1/2}\\
&\leq K\left\{I(\sigma_v^2<N)+(\sigma_v^2)^{-(m-q)/2}I(\sigma_v^2>N)\prod_{i\notin\{i_1,\ldots,i_q\}}(1-\rho\nu_i)\right\}\\
&\leq K\left\{I(\sigma_v^2<N)+(\sigma_v^2)^{-(m-q)/2}I(\sigma_v^2>N)\right\}, \quad \text{(A.13)}\end{aligned}$$

where we use the fact that $-1<\rho<1$ and $-1\leq\nu_i\leq1$ to claim $0<1-\rho\nu_i<2$. From (A.5) and (A.13), it follows by proceeding along the lines we did for the CAR model that the desired integral $\int f(\mathbf{y}_{(2)},\boldsymbol{\theta},\boldsymbol{\beta},\sigma_v^2,\rho)\,d\boldsymbol{\theta}d\boldsymbol{\beta}d\sigma_v^2d\rho$ is finite under the conditions of the theorem.

A.3 Details for the CAR Model

We now consider $k=4$ for the IAR model where

$$\boldsymbol{\Omega}_4=\mathbf{L}-\rho\mathbf{W}=\mathbf{L}^{1/2}(\mathbf{I}_m-\rho\mathbf{W}_*)\mathbf{L}^{1/2}.$$

Let $\boldsymbol{\Sigma}_4=\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_4^{-1}=\mathbf{L}^{-1/2}\left\{\delta\mathbf{L}+\sigma_v^2(\mathbf{I}_m-\rho\mathbf{W}_*)^{-1}\right\}\mathbf{L}^{-1/2}$. Then

$$\left|\boldsymbol{\Sigma}_4\right| \geq \left|\mathbf{L}\right|^{-1} k_*^m \prod_{i=1}^m \left\{\delta + \sigma_v^2 (1-\rho\nu_i)^{-1}\right\}, \tag{A.14}$$

where $k_*^m = \min\{l_{(1)}, 1\}$. Proceeding along the same line as in (A.11), we get that

$$(\mathbf{d}-\mathbf{G}\boldsymbol{\phi})^\top \boldsymbol{\Sigma}_4^{-1} (\mathbf{d}-\mathbf{G}\boldsymbol{\phi}) \geq \sum_{k=1}^q \frac{(\tilde{r}_{i_k} - \tilde{s}_{i_k}^\top \boldsymbol{\phi})^2}{\delta + \sigma_v^2 (1-\rho\nu_{i_k})^{-1}}. \tag{A.15}$$

Again, as we had for the two previous cases, we can use (A.14) and (A.15) to establish that the desired integral is finite under the conditions stated in the theorem.

A.4 Details for the LCAR Model

Finally, we consider $k=5$, where for the LCAR case we have

$$\boldsymbol{\Omega}_5 = \rho\mathbf{R} + (1-\rho)\,\mathbf{I}_m.$$

Suppose $r_1, \ldots, r_m$ are the eigenvalues of $\mathbf{R}$ and $\mathbf{P}_\mathbf{R}$ is an orthogonal matrix such that $\mathbf{P}_\mathbf{R}^\top \mathbf{R}\mathbf{P}_\mathbf{R} = \operatorname{diag}\{r_i\}_{i=1}^m$. Since $\mathbf{R}$ is a non-negative definite matrix, $r_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m r_i = \operatorname{tr}\mathbf{R} = \sum_{i=1}^m l_i$, implying that $r_1, \ldots, r_m$ are all bounded between 0 and $l = \sum_{i=1}^m l_i$. Then we can write

$$\boldsymbol{\Omega}_5 = \mathbf{P}_\mathbf{R} \left\{\operatorname{diag}\{\rho r_i + 1 - \rho\}_{i=1}^m\right\} \mathbf{P}_\mathbf{R}^\top,$$

and claim that for $0 < \rho < 1$, the eigenvalues of $\boldsymbol{\Omega}_5$ are all positive and bounded above by $\sum_{i=1}^m r_i + 1 = l + 1$. Then, with $\tilde{\mathbf{r}} = \mathbf{P}_\mathbf{R}^\top \mathbf{d}$, and $\tilde{\mathbf{S}} = \mathbf{P}_\mathbf{R}^\top \mathbf{G}$, we can establish an inequality similar to (A.11). Note that the nonsingular matrix $\tilde{\mathbf{S}}_1$ is a submatrix of $\tilde{\mathbf{S}}$ and is free from ρ. Boundedness of the eigenvalues of $\boldsymbol{\Omega}_5$ will lead to an inequality similar to (A.14). Finally, we get the desired integral is finite under the conditions of the theorem.

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