Catalogue no. 12-001-X
ISSN 1492-0921

Survey Methodology

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Release date: June 22, 2016

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0^s value rounded to 0 (zero) where there is a meaningful distinction between true zero and the value that was rounded
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^{r} revised
x suppressed to meet the confidentiality requirements of the *Statistics Act*
^{E} use with caution
F too unreliable to be published
* significantly different from reference category ($p < 0.05$)

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Survey Methodology, June 2016
Vol. 42, No. 1, pp. 179-187
Statistics Canada, Catalogue No. 12-001-X

A short note on quantile and expectile estimation in unequal probability samples

Linda Schulze Waltrup and Göran Kauermann[1]

Abstract

The estimation of quantiles is an important topic not only in the regression framework, but also in sampling theory. A natural alternative or addition to quantiles are expectiles. Expectiles as a generalization of the mean have become popular during the last years as they not only give a more detailed picture of the data than the ordinary mean, but also can serve as a basis to calculate quantiles by using their close relationship. We show, how to estimate expectiles under sampling with unequal probabilities and how expectiles can be used to estimate the distribution function. The resulting fitted distribution function estimator can be inverted leading to quantile estimates. We run a simulation study to investigate and compare the efficiency of the expectile based estimator.

Key Words: Quantiles; Expectiles; Probability proportional to size; Design-based; Auxiliary variable; Distribution function.

1 Introduction

Quantile estimation and quantile regression have seen a number of new developments in recent years with Koenker (2005) as a central reference. The principle idea is thereby to estimate an inverted cumulative distribution function, generally called the quantile function $Q(\alpha) = F^{-1}(\alpha)$ for $\alpha \in (0,1)$, where the 0.5 quantile $Q(0.5)$, the median, plays a central role. For survey data tracing from an unequal probability sample with known probabilities of inclusion Kuk (1988) shows how to estimate quantiles taking the inclusion probabilities into account. The central idea is to estimate a distribution function of the variable of interest and invert this in a second step to obtain the quantile function. Chambers and Dunstan (1986) propose a model-based estimator of the distribution function. Rao, Kovar and Mantel (1990) propose a design-based estimator of the cumulative distribution function using auxiliary information. Bayesian approaches in this direction have recently been proposed in Chen, Elliott, and Little (2010) and Chen, Elliott, and Little (2012).

Quantile estimation results from minimizing an L_1 loss function as demonstrated in Koenker (2005). If the L_1 loss is replaced by the L_2 loss function one obtains so called expectiles as introduced in Aigner, Amemiya and Poirier (1976) or Newey and Powell (1987). For $\alpha \in (0,1)$, this leads to the expectile function $M(\alpha)$ which, like the quantile function $Q(\alpha)$, uniquely defines the cumulative distribution function $F(y)$. Expectiles are relatively easy to estimate and they have recently gained some interest, see e.g., Schnabel and Eilers (2009), Pratesi, Ranalli, and Salvati (2009), Sobotka and Kneib (2012) and Guo and Härdle (2013). However since expectiles lack a simple interpretation their acceptance and usage in statistics is less developed than quantiles, see Kneib (2013). Quantiles and expectiles are connected in that a unique and invertible transformation function $h_y : [0,1] \to [0,1]$ exists so that $M(h(\alpha)) = Q(\alpha)$, see Yao and Tong (1996) and De Rossi and Harvey (2009). This connection can be used to estimate quantiles

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from a set of fitted expectiles. The idea has been used in Schulze Waltrup, Sobotka, Kneib and Kauermann (2014) and the authors show empirically that the resulting quantiles can be more efficient than empirical quantiles, even if a smoothing step is applied to the latter (see Jones 1992). An intuitive explanation for this is that expectiles account for all the data while quantiles based on the empirical distribution function only take the left (or the right) hand side of the data into account. That is, the median is defined by the 50% left (or 50% right) part of the data while the mean (as 50% expectile) is a function of all data points. In this note we extend these findings and demonstrate how expectiles can be estimated for unequal probability samples and how to obtain a fitted distribution function from fitted expectiles.

The paper is organized as follows. In Section 2 we give the necessary notation and discuss quantile regression in unequal probability sampling. This is extended in Section 3 towards expectile estimation. Section 4 utilizes the connection between expectiles and quantiles and demonstrates how to derive quantiles from fitted expectiles. Section 5 demonstrates in simulations the efficiency gain in quantiles derived from expectiles and a discussion concludes the paper in Section 6.

2 Quantile estimation

We consider a finite population with N elements and a continuous survey variable Y. We are interested in quantiles of the cumulative distribution function $F(y) = \sum_{i=1}^{N} 1\{Y_i \leq y\}/N$ and define as

$$Q(\alpha) = \inf\left\{\arg\min_q \sum_{i=1}^{N} w_\alpha(Y_i - q)|Y_i - q|\right\} \tag{2.1}$$

the Quantile function of Y (see Koenker 2005), where

$$w_\alpha(\varepsilon) = \begin{cases} \alpha & \text{for } \varepsilon > 0 \\ 1-\alpha & \text{for } \varepsilon \leq 0. \end{cases}$$

The "inf" argument in (2.1) is required in finite populations since the "arg min" is not unique. We draw a sample from the population with known inclusion probabilities π_i, $i = 1, \ldots, N$. Denoting by $y_1, \ldots, y_n$ the resulting sample, we estimate the quantile function by replacing (2.1) through its weighted sample version

$$\hat{Q}_N(\alpha) = \inf\left\{\arg\min_q \sum_{j=1}^{n} \frac{1}{\pi_j} w_{\alpha,j} |y_j - q|\right\} \tag{2.2}$$

with $w_{\alpha,j} = w_\alpha(y_j - q)$ as defined above. It is easy to see that the sum in (2.2) is a design-unbiased estimate for the sum in $Q(\alpha)$ given in (2.1). Nonetheless, because we take the "arg min" it follows that $\hat{Q}_N(\alpha)$ is not unbiased for $Q(\alpha)$. We therefore look at consistency statements for $\hat{Q}_N(\alpha)$ as follows. Let $R_i(q) = w_\alpha(y_i - q)|y_i - q|$ and

$$\bar{R}_N(q) := \frac{1}{N}\sum_i R_i(q).$$

We draw a sample from $R_i(q), i = 1, \ldots, N$ and assume we apply a consistent sampling scheme in that

$$\bar{r}_n(q) := \frac{1}{N}\sum_{j=1}^{n}\frac{1}{\pi_j} r_j(q)$$

is design-consistent for $\bar{R}_N(q)$, where $r_j(q)$ denotes the sample of $R_i(q)$. Note that $r_j(q)$ and hence $\bar{r}_n(q)$, $R_i(q)$ and $\bar{R}_N(q)$ also depend on α which is suppressed in the notation for readability. Let q_0 be the minimum of $\bar{R}_N(q)$ which is not necessarily unique due to the finite structure of the population. We can take the "inf" argument, i.e., $q_0 = \inf\{\arg\min \bar{R}_N(q)\}$, but for simplicity we assume a superpopulation model (see Isaki and Fuller 1982) by considering the finite population to be a sample from an infinite superpopulation. In the latter we assume that survey variable Y has a continuous cumulative distribution function so q_0 results in a unique α quantile. We get for $\delta > 0$

$$P(\bar{r}_n(q_0) < \bar{r}_n(q_0 - \delta)) \Leftrightarrow P\left(\frac{1}{N}\sum_{j=1}^{n}\frac{1}{\pi_j}\{r_j(q_0) - r_j(q_0 - \delta)\} < 0\right).$$

Note that the argument in the probability statement is a design-consistent estimate for $\bar{R}_N(q_0) - \bar{R}_N(q_0 - \delta)$, which is less than zero since q_0 is the minimum of $\bar{R}_N(\cdot)$. Hence, the probability tends to one in the sense of design consistency defined in Isaki and Fuller (1982). The same holds of course for $\delta < 0$. With this statement we may conclude that the estimated minimum $\hat{q}_0 = \arg\min \sum_{j=1}^{n} 1/\pi_j \; r_j(q)$ is a design-consistent estimate for q_0 so that $\hat{Q}_N(\alpha)$ in (2.2) is in turn design-consistent for $Q_N(\alpha)$. It is easily shown that $\hat{Q}_N(\alpha)$ is the inverse of the normed weighted cumulative distribution function

$$\hat{F}_N(y) := \frac{\sum_{j=1}^{n} 1\{y_j \le y\}/\pi_j}{\sum_{j=1}^{n} 1/\pi_j}$$

using the same notation as in Kuk (1988). Note that $\hat{F}_N(y)$ is the Hajek (1971) estimate of the cumulative distribution function (see also Rao and Wu 2009) and as such not a Horvitz-Thompson estimate. As a consequence $\hat{Q}_N(\alpha)$ is not design-unbiased. Nonetheless, $\hat{F}_N(y)$ is a valid distribution function, and hence it can be considered as normalized version of the Lahiri or Horvitz-Thompson estimator of the distribution function (see Lahiri 1951) which is denoted by

$$\hat{F}_L(y) := \frac{1}{N}\sum_{j=1}^{n} 1/\pi_j 1\{y_j \le y\}.$$

Kuk (1988) proposes to replace $\hat{F}_L(\cdot)$ with alternative estimates of the distribution function: Instead of estimating the distribution function itself he suggests to estimate the complementary proportion $\hat{S}_R(y)$ which then leads to the estimate $\hat{F}_R(y)$ defined through

$$\hat{F}_R(y) = 1 - \hat{S}_R(y) = 1 - \frac{1}{N}\sum_{j=1}^{n} 1/\pi_j 1\{y_j > y\}.$$

Resulting directly from these definitions we can express $\hat{F}_R(\cdot)$ in terms of $\hat{F}_N(\cdot)$ through

$$\hat{F}_R = 1 - \frac{1}{N}\sum_{j=1}^{n} 1/\pi_j + \hat{F}_L \quad \text{and} \quad \hat{F}_L = \frac{\sum_{j=1}^{n} 1/\pi_j}{N}\hat{F}_N. \tag{2.3}$$

Kuk (1988) shows that, under sampling with unequal probabilities, estimation of the median derived from $\hat{F}_R$ outperforms median estimates derived from $\hat{F}_N$ and $\hat{F}_L$ in terms of mean squared estimation error. Note that the estimators $\hat{F}_N$, $\hat{F}_L$ and $\hat{F}_R$ coincide in the case of simple random sampling without replacement where $\pi_j = \pi = n/N$.

3 Expectile estimation

An alternative to quantiles are expectiles. The expectile function $M(\alpha)$ is thereby defined by replacing the L_1 loss in (2.1) by the L_2 loss leading to

$$M(\alpha) = \arg\min_m \left\{\sum_{i=1}^{N} w_\alpha (Y_i - m)(Y_i - m)^2\right\}. \tag{3.1}$$

Note that $M(\alpha)$ is continuous in α even for finite populations. Moreover $M(0.5)$ equals the mean value $\bar{Y} = \sum_{i=1}^{N} Y_i/N$. Using the sample $y_1, \ldots, y_n$ with inclusion probabilities $\pi_1, \ldots, \pi_n$ we can estimate $M(\alpha)$ by replacing the sum in (2.2) by its sample version, i.e.,

$$\hat{M}(\alpha) = \arg\min_m \left\{\sum_{j=1}^{n} \frac{1}{\pi_j} w_{\alpha,j} \left(y_j - m\right)^2\right\}$$

with $w_{\alpha,j}$ as defined above. It is easy to see that the sum in $\hat{M}(\alpha)$ is a design-unbiased estimate for the sum in $M(\alpha)$. The estimate itself is however not design-unbiased like for the quantile function above. However the same arguments as for $Q_N(\alpha)$ in (2.2) may be used to establish design-consistency.

4 From expectiles to the distribution function

Both, the quantile function $Q(\alpha)$ and the expectile function $M(\alpha)$ uniquely define a distribution function $F(.)$. While $Q(\alpha)$ is just the inversion of $F(.)$ the relation between $M(\alpha)$ and $F(.)$ is more complicated. Following Schnabel and Eilers (2009) and Yao and Tong (1996), we have the relation

$$M(\alpha) = \frac{(1-\alpha)G(M(\alpha)) + \alpha\{M(0.5) - G(M(\alpha))\}}{(1-\alpha)F(M(\alpha)) + \alpha\{1 - F(M(\alpha))\}}, \tag{4.1}$$

where $G(m)$ is the moment function defined through $G(m) = \sum_{i=1}^{N} Y_i 1\{Y_i \le m\}/N$. Expression (4.1) gives the unique relation of function $M(\alpha)$ to the distribution function $F(.)$. The idea is now to solve (4.1) for $F(.)$, that is to express the distribution $F(.)$ in terms of the expectile function $M(.)$. Apparently, this is not possible in analytic form but it may be calculated numerically. To do so, we evaluate the fitted function $\hat{M}(\alpha)$ at a dense set of values $0 < \alpha_1 < \alpha_2 \ldots < \alpha_L < 1$ and denote the fitted values as $\hat{m}_l = \hat{M}(\alpha_l)$. We also define left and right bounds through $\hat{m}_o = \hat{m}_1 - c_0$ and $\hat{m}_{L+1} = \hat{m}_L + c_{L+1}$, where c_0 and c_L are some constants to be defined by the user. For instance, one may set $c_0 = \hat{m}_2 - \hat{m}_1$ and $c_{L+1} = \hat{m}_L - \hat{m}_{L-1}$. By doing so we derive fitted values for the cumulative distribution function $F(.)$ at $\hat{m}_l$ which we write as $\hat{F}_l := \hat{F}(\hat{m}_l) = \sum_{j=1}^{l} \hat{\delta}_j$ for non-negative steps $\hat{\delta}_j \geq 0, j = 1, \ldots, L$ with $\sum_{j=1}^{L} \hat{\delta}_j \leq 1$. We define $\hat{\delta}_{L+1} = 1 - \sum_{l=1}^{L} \hat{\delta}_l$ to make $\hat{F}(.)$ a distribution function. Assuming a uniform distribution between the dense supporting points $\hat{m}_l$ we may express the moment function $G(.)$ by simple stepwise integration as

$$\hat{G}_l := \hat{G}(\hat{m}_l) = \int_{-\infty}^{m_l} x \, d\hat{F}(x) = \sum_{j=1}^{l} \hat{d}_j \hat{\delta}_l,$$

where $\hat{d}_j = (\hat{m}_j - \hat{m}_{j-1})/2$ with the constraint that $\hat{G}_{L+1} = \hat{M}(0.5)$ and $\hat{M}(0.5) = \sum_{j=1}^{n} (y_j / \pi_j) \Big/ \sum_{j=1}^{n} (1/\pi_j)$. With the steps $\hat{\delta}_l, l = 1, \ldots, L$ we can now re-express (4.1) as

$$\hat{m}_l = \frac{(1-\alpha)\sum_{j=1}^{l} \hat{d}_j \hat{\delta}_j + \alpha\left(\hat{M}(0.5) - \sum_{j=1}^{l} \hat{d}_j \hat{\delta}_j\right)}{(1-\alpha)\sum_{j=1}^{l} \hat{\delta}_j + \alpha\left(1 - \sum_{j=1}^{l} \hat{\delta}_j\right)}, \quad l = 1, \ldots, L,$$

which is then be solved for $\hat{\delta}_1, \ldots, \hat{\delta}_L$. This is a numerical exercise which is conceptually relatively straightforward. Details can be found in Schulze Waltrup et al. (2014). Once we have calculated $\hat{\delta}_1, \ldots, \hat{\delta}_L$ we have an estimate for the cumulative distribution function which is denoted as $\hat{F}_N^M(y) = \sum_{l:\hat{m}_l < y} \hat{\delta}_l$. We may also invert $\hat{F}_N^M(.)$ which leads to a fitted quantile function which we denote with $\hat{Q}_N^M(\alpha)$.

As Kuk (1988) shows, both theoretically and empirically, $\hat{F}_R(.)$ is more efficient than $\hat{F}_N(.)$. We make use of this relationship and apply it to $\hat{F}_N^M(.)$, which yields the estimator

$$\hat{F}_R^M := 1 - \frac{1}{N}\sum_{j=1}^{n} 1/\pi_j + \frac{\sum_{j=1}^{n} 1/\pi_j}{N} \hat{F}_N^M.$$

In the next section we compare the quantiles calculated from the expectile based estimator $\hat{F}_R^M$ with quantiles calculated from $\hat{F}_R$. Note that neither $\hat{F}_R^M$ nor $\hat{F}_R$ are proper distribution functions since they are not normed to take values between 0 and 1.

5 Simulations

We run a small simulation study to show the performance of the expectile based estimates. In the following, we make use of the Mizuno sampling method (see Midzuno 1952) and define the inclusion probabilities π_j proportional to a measure of size x, see R package “sampling” by Tillé and Matei (2015). We examine two data sets also used in Kuk (1988). The first data set (Dwellings) contains two variables, the number of dwelling units (X), and the number of rented units (Y), which are highly correlated (with a correlation of 0.97); see also Kish (1965). The second data set (Villages) includes information on the population (X) and on the number of workers in household industry (Y) for 128 villages in India; see Murthy (1967). In the second data set the correlation between Y and X is 0.54. In order to compare our simulation results with the results of Kuk (1988) we choose the same sample size of $n = 30$ (from a total population of $N = 270$ for the Dwellings data and $N = 128$ for the Villages data).

We compare quantiles defined by inversion of $\hat{F}_R$ with quantiles defined by inversion of $\hat{F}_R^M$. In Table 5.1 we give the root mean squared error (RMSE) and the relative efficiency for specified quantiles. We note that the median for the village data and for the Dwelling data also upper quantiles derived from expectiles yield increased efficiency. Also the efficiency gain does not hold uniformly as we observe a loss of efficiency for lower quantiles.

Table 5.1
Comparison of mean squared error on a basis of 500 replications

	α	quantiles $\sqrt{\text{MSE}\left(\hat{Q}_R(\alpha)\right)}$	quantiles from expectiles $\sqrt{\text{MSE}\left(\hat{Q}_R^M(\alpha)\right)}$	relative efficiency $\frac{\sqrt{\text{MSE}\left(\hat{Q}_R^M(\alpha)\right)}}{\sqrt{\text{MSE}\left(\hat{Q}_R(\alpha)\right)}}$
Dwellings	0.1	2.57	2.76	1.07
	0.25	1.77	1.97	1.11
	0.5	2.45	2.35	0.96
	0.75	3.15	2.91	0.92
	0.9	4.20	3.43	0.82
Villages	0.1	5.52	6.65	1.21
	0.25	11.41	10.31	0.90
	0.5	12.29	11.69	0.95
	0.75	16.24	15.41	0.95
	0.9	13.31	18.34	1.38

To obtain more insight we run a simulation scenario which involves a larger sample size of $n = 100$ selected from populations of sizes $N = 1{,}000$ and $N = 10{,}000$. We draw Y and X from a bivariate log standard normal distribution with $\mu = 0$ and $\sigma = 1$. The variables Y and X are drawn such that the correlation between the variables is equal to 0.9. We again calculate the root mean squared error for a range of α values and show the relative efficiency of the expectile based approach in Figure 5.1. For better visual presentation we show a smoothed version of the relative efficiency. We notice a reduction in the root mean squared error for both cases $N = 1{,}000$ and $N = 10{,}000$. We may conclude that the expectiles can easily be fitted in unequal probability sampling and the relation between expectiles and the distribution function can be used numerically to calculate quantiles with increased efficiency. This efficiency gain holds for upper quantiles only, that is for α bounded away from zero. Note however that the sampling scheme is such that large values of Y are sampled with higher probability, reflecting that the sampling scheme aims to get more reliable estimates for the right hand side of the distribution function, i.e., for large quantiles. If we are interested in small quantiles we should use a different samling scheme by giving individuals with small values of Y an increased inclusion probability. In this case the behavior shown in Figure 5.1 would be mirrored with respect to α.

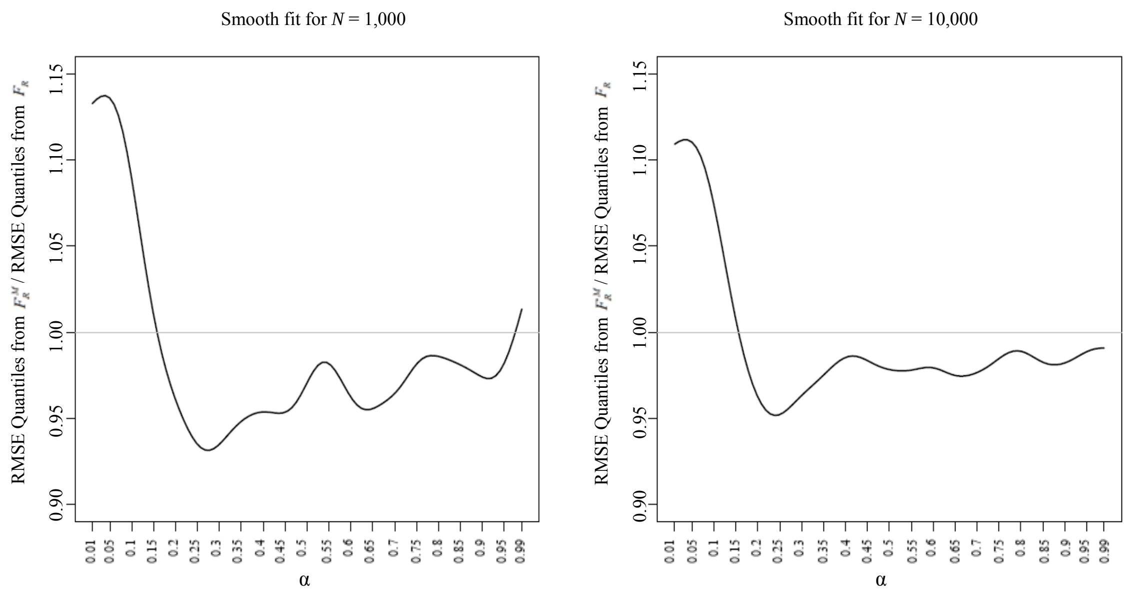

Figure 5.1 Relative Root Mean Squared Error (RMSE) of quantiles and quantiles from expectiles for the Probability Proportional to Size (PPS) design calculated from 500 repetitions (left: $N = 1{,}000$, right: $N = 10{,}000$).

6 Discussion

In Section 4 we extended the toolbox of expectiles to the estimation of distribution functions in the framework of unequal probability sampling. We defined expectiles for unequal probability samples. When comparing quantiles based on $\hat{F}_R$ with quantiles based on the expectile based estimator $\hat{F}_R^M$, we observed that the proposed estimator performs well in comparison to existing methods. The calculation of empirical expectiles is implemented in the open source software R (see R Core Team 2014) and can be found in the R-package expectreg by Sobotka, Schnabel, and Schulze Waltrup (2013). The calculation of the expectile based distribution function estimator $\hat{F}_N^M$ is also part of the R-package expectreg. The calculation of $\hat{F}_R^M$ is, however, more demanding as the calculation of $\hat{F}_R$ because it involves three steps: First, we calculate the weighted expectiles as described in Section 3; second, we estimate $\hat{F}_R^N$, and in a third step, we derive $\hat{F}_R^M$ from $\hat{F}_R^N$ (see Section 4). In the Log-Normal-Simulation it takes about 2-3 seconds for $N = 1{,}000$ to calculate $\hat{F}_R^M$ whereas the computational effort of $\hat{F}_R$ is barely noticeable.

Acknowledgements

Both authors acknowledge financial support provided by the Deutsche Forschungsgemeinschaft DFG (KA 1188/7-1).

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