On bias-robust mean squared error estimation for pseudo-linear small area estimators

by Ray Chambers, Hukum Chandra and Nikos Tzavidis

December 2011
On bias-robust mean squared error estimation for pseudo-linear small area estimators

Ray Chambers, Hukum Chandra and Nikos Tzavidis

Abstract

We propose a method of mean squared error (MSE) estimation for estimators of finite population domain means that can be expressed in pseudo-linear form, i.e., as weighted sums of sample values. In particular, it can be used for estimating the MSE of the empirical best linear unbiased predictor, the model-based direct estimator and the M-quantile predictor. The proposed method represents an extension of the ideas in Royall and Cumberland (1978) and leads to MSE estimators that are simpler to implement, and potentially more bias-robust, than those suggested in the small area literature. However, it should be noted that the MSE estimators defined using this method can also exhibit large variability when the area-specific sample sizes are very small. We illustrate the performance of the method through extensive model-based and design-based simulation, with the latter based on two realistic survey data sets containing small area information.

Key Words: Best linear unbiased prediction; M-quantile model; Model-based direct estimation; Random effects model; Small area estimation.

1. Introduction

Linear models, and linear predictors based on these models, are widely used in survey-based inference. However, such models run the risk of misspecification, particularly with regard to second order and higher moments. Bias-robust methods for estimating the mean squared error (MSE) of linear predictors of finite population quantities, i.e., methods that remain approximately unbiased under failure of assumptions about second order and higher moments, have been developed. Valliant, Dorfman and Royall (2000, Chapter 5) discuss bias-robust MSE estimation for such predictors when a population is assumed to follow a linear model.

In this paper we address a subsidiary problem, which is that of bias-robust MSE estimation for estimators of finite population domain means that can be expressed in pseudo-linear form, i.e., as weighted sums, but where the weights can depend on the sample values of the variable of interest. An important application, and one that motivates our approach, is small area inference. Consequently from now on we use ‘area’ to refer to a domain of interest. Our approach represents an extension of the ideas in Royall and Cumberland (1978) and appears to lead to simpler to implement MSE estimators than those that have been suggested in the small area literature.

The structure of the paper is as follows. In section 2 we discuss MSE estimation under an area-specific linear model. That is, we focus on estimation of the conditional MSE. We then show how our approach can be used for estimating the MSE of three different small area linear predictors when they are expressed in pseudo-linear form, (a) the empirical best linear unbiased predictor or EBLUP (Henderson 1953); (b) the model-based direct estimator (MBDE) of Chandra and Chambers (2009); and (c) the M-quantile predictor (Chambers and Tzavidis 2006). In section 3 we present results from a series of simulation studies that illustrate the model-based and the design-based properties of our approach to MSE estimation. Finally, in section 4 we summarise our main findings. Throughout, we use either $i$ or $h$ to index the $D$ small areas of interest, and either $j$ or $k$ to index the distinct population units in these areas.

2. Bias-robust MSE estimation for pseudo-linear estimators

2.1 MSE estimation under an area-specific linear model

We consider the situation where we have a finite population of size $N$ from which a sample of size $n$ is drawn. We assume that this population consists of $D$ non-overlapping domains, each one of which contains sampled units, with small realised sample sizes in each of the sampled domains. As noted earlier, and following standard practice, we refer to these domains as areas from now on. We assume also that there is a known number $N_i$ of population units in area $i$, with $n_i$ of these sampled. The total number of units in the population is $N = \sum_{i=1}^{D} N_i$, with corresponding total sample size $n = \sum_{i=1}^{D} n_i$. In what follows, we use $s$ to denote the collection of units in sample, with $s_i$ the subset drawn from area $i$, and use expressions like $j \in i$ and $j \in s$ to refer to the units making up area $i$ and sample $s$ respectively.

---

1. Ray Chambers, Centre for Statistical and Survey Methodology, University of Wollongong, Wollongong, NSW, 2522, Australia. E-mail: ray@uow.edu.au; Hukum Chandra, Indian Agricultural Statistics Research Institute, Library Avenue, New Delhi-110012, India. E-mail: hchandra@iasri.res.in; Nikos Tzavidis, Social Statistics and Southampton Statistical Sciences Research Institute, University of Southampton, Southampton, SO17 1BJ, UK. E-mail: n.tzavidis@soton.ac.uk.
Linear models are often used to motivate estimators for population means. However, when estimates are required for the corresponding area means, it is usually not realistic to assume that a linear model that applies to the population as a whole also applies within each area. We therefore adopt a conditional approach, and consider MSE estimation for estimators of area means when different linear models apply within different areas. In particular, we focus on estimators that can be expressed as weighted sums of the sample values, referring to them as ‘linear’ in what follows to indicate that they have a linear structure.

To start, let $y_j$ denote the value of $Y$ for unit $j$ of the population and suppose that this unit is in area $i$. We also assume an area-specific linear model for $y_j$ of the form

$$y_j = x_j' \beta_i + e_j, \quad (1)$$

Here $x_j$ is a $p \times 1$ vector of unit level auxiliary variables for unit $j$, $\beta_i$ is a $p \times 1$ vector of area-specific regression coefficients and $e_j$ is a unit level random effect with mean zero and variance $\sigma_j^2$ that is uncorrelated between different population units. We do not make any assumptions about $\sigma_j^2$ at this point. Note that throughout this paper we assume that the sampling method used is non-informative for the population units. We do not make any assumptions about zero and variance $\sigma_j^2$ of the auxiliary variables in area

$$\text{Var}(y_j - \hat{\mu}_j) = \sigma_j^2 \{ (1 - \phi_j)^2 + \sum_{k \in s(-j)} \phi_j \sigma_k^2 \} \quad (4)$$

and so

$$\text{Var}(y_j - \hat{\mu}_j) = \sigma_j^2 \{ (1 - \phi_j)^2 + \sum_{k \in s(-j)} \phi_j \sigma_k^2 \} \quad (4)$$

under (1). Here $s(-j)$ denotes the sample $s$ with unit $j$ excluded. If in addition $\hat{\mu}_j$ is unbiased for $\mu_j$ under (1), i.e.,

$$E(y_j - \hat{\mu}_j) = 0, \quad (5)$$

we can then adopt the approach of Royall and Cumberland (1978) and estimate (3) by

$$\hat{\lambda}_j = (1 - \phi_j)^2 + \sum_{k \in s(-j)} \phi_j \sigma_k^2$$

usually, the estimates $\hat{\sigma}_j^2$ of the residual variances in (6) are derived under a ‘working model’ refinement to (1). In the situation of most concern to us, where the sample sizes within the different areas are too small to reliably estimate area-specific variability, a pooling assumption can be made, i.e., $\sigma_j^2 = \sigma^2$, in which case we put

$$\hat{\sigma}_j^2 = \hat{\sigma}_j^2 = n^{-1} \sum_{k \in s(-j)} (1 - \phi_j)^2 + \sum_{k \in s(-j)} \phi_j \sigma_k^2 (y_j - \hat{\mu}_j)^2.$$
\[ \hat{M}(\tilde{m}_i) = \hat{V}(\tilde{m}_i) + \hat{B}^2(\tilde{m}_i) - \hat{V}(\hat{B}(\tilde{m}_i)), \]  

(10)

where \( \hat{V}(\hat{B}(\tilde{m}_i)) \) is a suitable estimator of the variance of (9). However, we do not recommend use of (10). To see this, let \( B = D^{-1}\sum_{h=1}^{D} \hat{\beta}_h \) and put \( d_h = \hat{\beta}_h - B \), where \( \hat{\beta}_h \) is the estimator of \( \beta_h \) implied by the weights \( \phi_{ij} \). Furthermore, put \( w_{hi} = \sum_{j \in i} w_{ij} \) and \( x_{whi} = w_{ij} \sum_{j \in i} w_{ij} x_j \), so \( x_{whi} = \sum_{h=1}^{D} \sum_{j \in i} w_{ij} x_j = \sum_{h=1}^{D} w_{hi} x_{whi} \) is the estimate of \( x_i \) based on the weights \( w_{hi} \). Finally, let \( \delta_{hi} = x_{whi}^T d_h - x_{whi}^T d_i \) and put \( \delta_i = \sum_{h=1}^{D} w_{hi} \delta_{hi} \). Then (9) can be written

\[ \hat{B}(\tilde{m}_i) = (x_{whi} - \bar{x}_i)^T B \]

\[ + \left( \sum_{h=1}^{D} w_{hi} (x_{whi} - \bar{x}_i)^T d_h - x_{whi}^T d_i \right) \]

\[ = (x_{whi} - \bar{x}_i)^T B \]

\[ + \left( \sum_{h=1}^{D} w_{hi} (x_{whi} - \bar{x}_i)^T d_h + \sum_{h=1}^{D} w_{hi} x_{whi}^T d_h - x_{whi}^T d_i \right) \]

\[ = (x_{whi} - \bar{x}_i)^T B \]

\[ + \sum_{h=1}^{D} w_{hi} (x_{whi} - \bar{x}_i) (x_{whi} - \bar{x}_i)^T d_h + \delta_i. \]  

(11)

Typically, \( D \) will be large and the leading term in the variance of (9) will be the variance of \( \delta_i \) in (11). If this leading term is large, then \( \hat{V}(\hat{B}(\tilde{m}_i)) \) will also be large, and (10) could take negative values. We therefore recommend that (8), rather than (10), be used. An immediate consequence is that (8) is then a conservative estimator of the MSE of \( \hat{m}_i \) under (1). This may be acceptable provided that the variance of \( \delta_i \) is small. However, for very small values of \( n_i \), this variance can be large, causing (8) to substantially overestimate the actual MSE of \( \hat{m}_i \). We therefore recommend a preliminary empirical assessment of the size of the variance of \( \delta_i \) relative to the value of (7) in this situation. If this assessment indicates that the variance of \( \delta_i \) dominates (7), then (8) should not be used.

2.2 MSE estimation for pseudo-linear small area estimators

The approach to conditional MSE estimation outlined in the previous sub-section assumed that the weights defining the linear estimator \( \hat{m}_i \) do not depend on the sample values of \( Y \). However, most small area estimators do not satisfy this condition, in the sense that they are pseudo-linear in structure, with weights that do depend on these sample values. For example, the Best Linear Unbiased Predictor (BLUP) of \( m_i \) under the linear mixed model variant of (1) where the area-specific regression parameters \( \beta_i \) are independent and identically distributed realisations of a random variable with expected value \( \beta \) and covariance matrix \( \Gamma \), can be written as a weighted sum of the sample values of \( Y \) where the weights depend on \( \Gamma \) (see Royall 1976). Consequently, the empirical version of this predictor, the widely used EBLUP, is computed by substituting an efficient sample estimate of \( \Gamma \) (e.g., the REML estimate) into the BLUP weights. If the linear mixed model assumption is true, this sample estimator of \( \Gamma \) converges to the true value and consequently the EBLUP weights converge to the BLUP weights. That is, for large values of the overall sample size \( n \), we can treat the EBLUP weights as fixed and use the MSE estimator (8) for the EBLUP. Of course, the EBLUP weights are not really fixed, and so (8) is therefore an approximation to the true MSE of the EBLUP that ignores the contribution to this MSE arising from the variability in estimation of \( \Gamma \). However, this potential underestimation needs to be balanced against the bias robustness of (8) under misspecification of the second order moments of \( Y \).

An important advantage of (8) is that it can be used with a range of small area estimators that can be expressed in pseudo-linear form. In particular, many small area estimators developed under models that are variants of (1) can be written in this form, i.e., as weighted sums of the sample values of \( Y \). To illustrate, we now focus on three such estimators: the EBLUP (Rao 2003, Chapter 6), the Model-Based Direct Estimator (MBDE) of Chandra and Chambers (2009) and the M-quantile predictor of Chambers and Tzavidis (2006). Each of these estimators can be written in pseudo-linear form, with weights that satisfy \( w_{ij} = O(n_i^{-1/2}) \) for \( j \in s_i \) and \( w_{ij} = o(n_i^{-1}) \) for \( j \notin s_i \), and so (8) can be used.

2.2.1 MSE estimation for the EBLUP

We first consider the well-known EBLUP for \( m_i \) based on a unit level linear mixed model extension of (1) of the form

\[ y_i = X_i \beta + Z_i u_i + e_i \]

(12)

where \( y_i \) is the \( N_i \)-vector of population values of \( y_j \) in area \( i \), \( X_i \) is the corresponding \( N_i \times p \) matrix of auxiliary variable values \( x_j \), \( Z_i \) is the \( N_i \times q \) component of \( X \), corresponding to the \( q \)-random components of \( \beta \), \( u_i \) is the associated \( q \)-vector of area-specific random effects and \( e_i \) is the \( N_i \)-vector of individual random effects. It is typically assumed that the area and individual effects are mutually independent, with the area effects independently and identically distributed as \( N(0, \Omega) \) and the individual effects independently and identically distributed as \( N(0, \sigma^2) \). See Rao (2003, Chapter 6) for development of the underlying theory of this predictor. We note that the EBLUP can be written in pseudo-linear form,

\[ m_i^{EBLUP} = \sum_{j \in s_i} w_{ij} y_j \]

(13)
where
\[ w_{ij}^{\text{EBLUP}} = \left( w_{ij}^{\text{EBLUP}} \right) \]
\[ = N_n^{-1} \left\{ \Delta_i + \left\{ \hat{H}_i^T \Sigma_{rr}^{-1} (I_n - \hat{H}_i \Sigma_{rr}^{-1} \hat{H}_i^T) \Sigma_{rr}^{-1} \Sigma_{cs} \right\} \Delta_c \right\}. \]

Here \( \Delta_c \) is the vector of size \( N - n \) that ‘picks out’ the non-sampled units in area \( i \), \( X_s \) and \( X_r \) are the matrices of order \( n \times p \) and \( (N - n) \times p \) respectively of the sample and non-sample values of the auxiliary variables, \( I_n \) is the identity matrix of order \( n \), \( \hat{H}_i = (X_i \Sigma_{rr}^{-1} X_i^T)^{-1} X_i \Sigma_{rr}^{-1} \Sigma_{cs} \) is defined immediately before (11) takes the form
\[ \Delta_c = \hat{H}_i^T \Sigma_{rr}^{-1} + \text{diag} \left\{ \mathbf{Z}_i \hat{\Omega} \mathbf{Z}_i^T \right\} \text{ where } \hat{\Omega} = \text{diag} \left\{ \mathbf{Z}_i \hat{\Omega} \mathbf{Z}_i^T \right\}, \]
i.e., \( \hat{\Omega} = \text{diag} \left\{ \mathbf{Z}_i \hat{\Omega} \mathbf{Z}_i^T \right\} \) component of \( \mathbf{Z}_i \) and \( \hat{\Omega} \) are suitable (e.g., ML or REML) estimates of the variance components of (12).

Given this setup, estimation of the conditional MSE of the EBLUP can be carried out using (8) with weights defined following (13). In turn, this requires that we have access to unbiased estimators \( \hat{\mu}_j \) of the area specific individual expected values \( \mu_j \). However, such estimators may be unstable when area sample sizes are small. Consequently, it is tempting to replace \( \hat{\mu}_j \) by the EBLUP for \( y_j \), i.e.,
\[ y_j^{\text{EBLUP}} = x_j^T \hat{\beta}^{\text{EBLUP}} + \delta_j \hat{\mu}_j = \hat{y}_j^{\text{EBLUP}} \]
develops the Empirical Best Linear Unbiased Estimator of \( \beta \) in the linear mixed model (12) and \( \hat{\mu}_j \) denotes the predicted area effect for the area \( i \) that ‘picks out’ the \( \mu_j \)’s.

Finally, we observe that when (14) is used in (8), the estimated bias (9) becomes
\[ \hat{B}(\hat{m}_j) = \sum_{h=1}^{D} \left( \sum_{i=1}^{c} w_{ij}^{\text{EBLUP}} \right) \mathbf{z}_h \mathbf{u}_h - \mathbf{z}_j^{T} \mathbf{u}_j \]
since the EBLUP weights (13) are ‘locally calibrated’ on \( X \), i.e., \( \sum_{i=1}^{c} w_{ij}^{\text{EBLUP}} x_j = \mathbf{x}_j \). It follows that in this case the variable \( \delta_j \) defined immediately before (11) takes the form
\[ \delta_j = \sum_{h=1}^{D} w_{hi}^{\text{EBLUP}} \mathbf{z}_h \mathbf{u}_h - \mathbf{z}_j^{T} \mathbf{u}_j \]
and \( \delta_j \) can be approximated by
\[ \delta_j \approx \sum_{h=1}^{D} w_{hi}^{\text{EBLUP}} \mathbf{z}_h \mathbf{u}_h - \mathbf{z}_j^{T} \mathbf{u}_j \]
where \( w_{hi}^{\text{EBLUP}} = \sum_{j=1}^{c} \mathbf{w}_{ij}^{\text{EBLUP}} \). For a large enough overall sample size \( \delta_j \) can therefore be estimated via
\[ \hat{V}(\hat{\delta}_j) = \sum_{h=1}^{D} \left( \mathbf{w}_h^{\text{EBLUP}} \right)^2 \mathbf{z}_h \mathbf{z}_h + \mathbf{\Omega} + \mathbf{\hat{\Omega}} \mathbf{Z}_h \mathbf{Z}_h^{T} \mathbf{z}_h, \]
where \( \mathbf{\Omega} \) is the BLUP equivalent of \( \mathbf{w}_h^{\text{EBLUP}} \). The variance of \( \delta_j \) can therefore be estimated via
\[ \hat{V}(\hat{\delta}_j) = \sum_{h=1}^{D} \left( \mathbf{w}_h^{\text{EBLUP}} \right)^2 \mathbf{z}_h \mathbf{z}_h + \mathbf{\hat{\Omega}} + \mathbf{\hat{\Omega}} \mathbf{Z}_h \mathbf{Z}_h^{T} \mathbf{z}_h, \]

If \( \hat{V}(\hat{\delta}_j) \) is small relative to the value of (7) in this case, then (8) can be used to estimate the MSE of the EBLUP. However, when \( \eta_j \) is very small, this condition may not hold. In such cases it may be advisable to consider more
model-dependent MSE estimators like the Prasad-Rao (PR) MSE estimator (Prasad and Rao 1990; Rao 2003, section 7.2.3). When a random means model is assumed, but the between area variability is very small relative to the within area variability, this advice extends to moderate area sample sizes as we now show.

### 2.2.2 MSE estimation for the EBLUP under the random means model

The random means model is the special case of (12) where \( y_j = \beta + u_j + e_j \), with \( u_j \sim N(0, \sigma_u^2) \) and \( e_j \sim N(0, \sigma_e^2) \). The EBLUE of \( \beta \) is then \( \hat{\beta} = \sum_{h=1}^{D} \hat{\alpha}_h \bar{y}_h \) with \( \hat{\alpha}_h = (\hat{\phi} + n_h^{-1})^{-1} (\sum_{h=1}^{D} (\hat{\phi} + n_h^{-1})^{-1} \) and \( \hat{\phi} = \hat{\sigma}_u^2 / \hat{\sigma}_e^2 \), and the EBLUP (13) is defined by weights of the form

\[
w_{ij}^{EBLUP} = (1 - f_i)(1 - \hat{\gamma}_i) \sum_{h=1}^{D} \hat{\alpha}_h n_h^{-1} I(j \in h)
\]

with \( \hat{\gamma}_i = n_i \hat{\phi}(1 + n_i \hat{\phi})^{-1} \). For \( j \in h \), \( \mu_j = \sum_{k \in h} \phi_k y_k = \bar{y}_h \) and so

\[
\hat{\lambda}_j = (1 - \hat{\phi})^{-1} + \sum_{k \in h(j)} \hat{\phi}_k
\]

\[
\hat{\lambda}_j = (1 - n_h^{-1})^{-1} + (n_h - 1)n_h^{-2} = (n_h - 1)n_h^{-1}.
\]

It follows that the estimator (7) of the conditional prediction variance of \( \hat{m}_i^{EBLUP} \) in this case is

\[
\hat{V}(\hat{m}_i^{EBLUP}) = (1 - f_i)^2 \left[ \sum_{h=1}^{D} ((1 - \hat{\gamma}_i)^2 \hat{\alpha}_h n_h^{-2} + (N_i - n_i)^{-1} n_i s_h^2 + \hat{\gamma}_i n_i^{-1} \{2(1 - \hat{\gamma}_i) \hat{\alpha}_i + \hat{\gamma}_i \} s_i^2 \right],
\]

where \( s_h^2 = (n_h - 1)^{-1} \sum_{j \in h} (y_j - \bar{y}_h)^2 \), while from (9) the estimator of the conditional prediction bias of \( \hat{m}_i^{EBLUP} \) is

\[
\hat{B}(\hat{m}_i^{EBLUP}) = (1 - f_i)(1 - \hat{\gamma}_i)(\hat{\beta} - \bar{y}_h).
\]

For \( h \neq i \) we also then have

\[
w_{hi}^{EBLUP} = \sum_{j \in h \setminus i} w_{ij}^{EBLUP} = (1 - f_i) \hat{\alpha}_h (1 + n_i \hat{\phi})^{-1} \approx \hat{\alpha}_h (1 + n_i \hat{\phi})^{-1}
\]

when we ignore \( O(N_i^{-1}) \) terms. A similar approximation to (15) therefore leads to

\[
\hat{V}(\hat{\delta}_i) = \sum_{h=1}^{D} (w_{hi}^{EBLUP})^2 \left( \hat{\sigma}_u^2 + n_h^{-1} \hat{\sigma}_e^2 \right)
\]

\[
\approx \hat{\sigma}_u^2 \sum_{h=1}^{D} \frac{\hat{\alpha}_h}{1 + n_i \hat{\phi}} \left( \frac{1 + m\hat{\phi}}{m} \right) n_h \approx n_i^{-1} \left( 1 + m\hat{\phi} \right)^{-1} \hat{\sigma}_u^2
\]

while the corresponding approximation to \( \hat{V}(\hat{m}_i^{EBLUP}) \) is

\[
\hat{V}(\hat{m}_i^{EBLUP}) \approx \sum_{h=1}^{D} (1 + m\hat{\phi})^{-2} D^{-1} n_h \approx (1 + m\hat{\phi})^{-2} \left( D^{-1} \sum_{h=1}^{D} n_h^2 \right) + m\hat{\phi}(2 + m\hat{\phi})^{-2}.
\]

Comparing these approximations to \( \hat{V}(\hat{\delta}_i) \) and \( \hat{V}(\hat{m}_i^{EBLUP}) \) we see that if \( m\hat{\phi} \) is small (e.g., when \( m \) and \( \hat{\phi} \) are both small) then these terms will be of similar magnitude. In this situation we expect (8) to overestimate the true MSE of the EBLUP. In particular, the approximation to (8) when \( m\hat{\phi} \) is small and \( N_i \) is large is

\[
\hat{M}(\hat{m}_i^{EBLUP}) \approx n_i^{-1} \left( D^{-1} \sum_{h=1}^{D} n_h^2 \right) + \left( \bar{y}_h - D^{-1} \sum_{h=1}^{D} \bar{y}_h \right)^2.
\]

Note that the expectation of the squared residual on the right hand side of (16) when \( m\hat{\phi} \) is small is \( (1 - D^{-1})(\hat{\sigma}_u^2 + m^{-1}\sigma_e^2) = O(1) \) and so it is the leading term in this estimator in this situation. This expression can be compared with the corresponding one for the MSE estimator of the EBLUP suggested by Prasad and Rao (1990). Under the random means model, the PR MSE estimator is

\[
\hat{M}_{PR}(\hat{m}_i^{EBLUP}) = (1 - f_i)^2 \left[ \hat{\gamma}_i m^{-1} \hat{\sigma}_u^2 + (1 - \hat{\gamma}_i) \left( m \sum_{k=1}^{D} \hat{\tau}_h^{-1} \right) + N_i^{-1}(1 - f_i)^{-1} \hat{\sigma}_u^{-2} \right]
\]

\[
+ \frac{2}{T} \sum_{j=1}^{m} \hat{\tau}_j \left[ \hat{\sigma}_u^{-2} \left( \frac{n - D}{\hat{\sigma}_u^2} + \sum_{k=1}^{D} \hat{\tau}_h^{-2} \right) \right]
\]

\[
+ \hat{\sigma}_u^2 m^2 \sum_{h=1}^{D} \hat{\tau}_h^{-2} + 2\hat{\sigma}_u^2 \hat{\sigma}_e^2 m \sum_{h=1}^{D} \hat{\tau}_h^{-2}
\]

where \( \hat{\tau}_j = n_j \hat{\sigma}_u^2 + \hat{\tau}_j \) and

\[
T = \frac{n - D}{\hat{\sigma}_u^2} \sum_{h=1}^{D} n_h \hat{\tau}_h^{-2}.
\]

Assuming \( n_i = m, m\hat{\phi} \) is small and \( N_i \) is large, \( \hat{M}_{PR}(\hat{m}_i^{EBLUP}) \) has the approximation

\[
\hat{M}_{PR}(\hat{m}_i^{EBLUP}) \approx n_i^{-1} \left( D^{-1} \sum_{h=1}^{D} n_h^2 \right) + \left( \bar{y}_h - D^{-1} \sum_{h=1}^{D} \bar{y}_h \right)^2.
\]
\[
\hat{m}_{\text{pr}}(\hat{m}^*_{\text{EBLUP}}) \approx \sigma^2 \{n^{-1} + 2(n - D)^{-1}\} + \sigma^2_s. \tag{17}
\]

Comparing (16) and (17) we can see that the instability and the overestimation associated with the use of (8) in this situation are both due to the use of the square of the single degree of freedom area level residual \(\bar{y}_{is} - D^{-1}\Sigma_{t=1}^D \gamma_{hs}\) as an estimator of \(\sigma^2_s\). This reinforces earlier comments that (8) should not generally be used for estimating the MSE of the EBLUP if the area sample sizes are very small or, in the special case of the random means model, for moderate area sample sizes when the between area variability is very small relative to the within area variability.

### 2.2.3 MSE estimation for the MBDE

The second predictor of \(m_i\) that we consider is the Model-Based Direct Estimator (MBDE) described in Chandra and Chambers (2009). This is based on the same linear mixed model (12) as the EBLUP, with the MBDE predictor defined as

\[
\tilde{m}_i^{\text{MBDE}} = \sum_{j \in s_i} w^{\text{MBDE}}_{ij} y_j = (w^{\text{MBDE}}_{is})^T y_s \tag{18}
\]

where

\[
w^{\text{MBDE}}_{ij} = \frac{I(j \in s_i) w^{\text{EBLUP}}_j}{\sum_{k \in s_i} I(k \in s_i) w^{\text{EBLUP}}_k}. \tag{19}
\]

Here \(I(j \in s_i)\) is the indicator function for unit \(j\) to be in the area \(i\) sample, and \(w^{\text{EBLUP}}_j = (w^{\text{EBLUP}}_i)^{-1}\) is the vector of weights that defines the EBLUP for the population total of the \(y_j\) under (12), i.e.,

\[
w^{\text{EBLUP}}_j = (w^{\text{EBLUP}}_i)^{-1} = 1_n + (H^{T}_s X^T_s + (I_n - \hat{H}^{T}_s X^T_s) \Sigma^{-1}_s \Sigma^{-1}_{ss}) 1_{N-n}, \tag{20}
\]

where \(1_n (1_{N-n})\) denotes the unit vector of size \(n\) (\(N-n\)) and \(\hat{H}_s\) was defined in section 2.2.1. In this case pseudo-linearisation based estimation of the area-specific MSE of the MBDE is carried out using (8), with weights defined by (19). Note that the estimated expected values used in (8) when applied to the MBDE are the same as the unshrunkened estimates (14) used with the EBLUP, reflecting the fact that both the MBDE and the EBLUP are based on the same linear mixed model (12). However, the MBDE weights (19) are not locally calibrated, and so the squared bias term in (8) cannot be ignored when estimating the MSE of this predictor. Furthermore, since \(w^{\text{MBDE}}_{hi} = \sum_{j \in s_h} w^{\text{MBDE}}_{ij} = 0\) for \(h \neq i\), we have \(\delta_i = 0\) for the MBDE and so the bias estimator (9) works well in this case.

### 2.2.4 MSE estimation for the M-quantile estimator

The third estimator that we consider is based on the M-quantile modelling approach described in Chambers and Tzavidis (2006). This approach does not assume an underlying linear mixed model, relying instead on characterising the relationship between \(y_j\) and \(x_j\) in area \(i\) in terms of the linear M-quantile model that best ‘fits’ the sample \(y_j\) values from this area. That is, this approach replaces (12) by a model of the form

\[
y_j = X_j \beta(q_i) + e_j \tag{21}
\]

where \(\beta(q)\) denotes the coefficient vector of a linear model for the regression M-quantile of order \(q\) for the population values of \(Y\) and \(X\), and \(q_i\) denotes the M-quantile coefficient of area \(i\). Given an estimate \(\hat{q}_i\) of \(q_i\), an iteratively reweighted least squares (IRLS) algorithm is used to calculate an estimate

\[
\hat{\beta}(\hat{q}_i) = \{X'_jW_j(\hat{q}_i)X'_j\}^{-1} X'_jW_j(\hat{q}_i)y_s \tag{22}
\]

of \(\beta(q)\) in (20), and a non-sample value of \(y_j\) in area \(i\) is then predicted by \(\hat{y}_j = X'_j\hat{\beta}(\hat{q}_i)\). Here \(W_j(\hat{q}_i)\) is the diagonal matrix of final weights used in the IRLS algorithm.

Tzavidis, Marchetti and Chambers (2010) note that value of the M-quantile estimator suggested in Chambers and Tzavidis (2006) can be interpreted as the expected value of \(Y\) in area \(i\) with respect to a biased estimator of the distribution function of this variable in the area. They therefore develop an improved M-quantile estimator, replacing this biased distribution function estimator by the Chambers and Dunstan (1986) distribution function estimator under the area-specific model (1). This corresponds to predicting \(m_i\) by

\[
\hat{m}_i^{\text{MQ}} = \sum_{j \in s_i} w^{\text{MQ}}_{ij} y_j = (w^{\text{MQ}}_{is})^T y_s \tag{23}
\]

where

\[
w^{\text{MQ}}_{is} = n_i^{-1} \Delta_{is} + (1 - N_i^{-1} n_i) W_j(\hat{q}_i)X_j \{X'^T_j W_j(\hat{q}_i)X_j\}^{-1} (\overline{x}_j - \overline{x}_i). \tag{24}
\]

Here \(\overline{x}_s\) and \(\overline{x}_i\) are the vectors of sample and non-sample means of the \(x_j\) in area \(i\). It is not difficult to show that the weights following (22) are locally calibrated. Furthermore, if we then put \(\hat{\lambda}_j = X'_j \hat{\beta}(\hat{q}_i)\), where \(\hat{\beta}(\hat{q}_i)\) is defined by (21), it is easy to see that (9) is zero and so the area-specific MSE of the bias-corrected M-quantile estimator (22) can be estimated using just the estimated prediction variance component (7). Since the constant \(\hat{\lambda}_j\) in (7) is typically very close to one under M-quantile estimation, we set it equal to this value whenever we compute values of (7) that relate to...
small area estimation (SAE) under the M-quantile modelling approach.

As we have already done with the EBLUP, we note that use of (7) implicitly treats the weights defining (22) as fixed, which is actually not the case since the matrix \( W_i(q_i) \) is a function of the sample values of \( Y \). An immediate consequence is that pseudo-linearisation based estimation of the MSE of the M-quantile predictor via (7) is a first order approximation to the true MSE of this estimator. Nevertheless, since accounting for weight variability in the definition of the M-quantile estimator considerably complicates estimation of its MSE - see Street, Carroll and Ruppert (1988) for an examination of this issue in the context of ‘standard’ M-estimation of regression coefficients - it is of interest to see how the relatively simple estimator (7) performs when used to estimate this MSE.

2.3 MSE estimation for the pseudo-linear synthetic EBLUP

In many SAE applications there are areas that contain no sample, and hence synthetic estimation is used. Although such estimators do not fit into the class of pseudo-linear estimators considered in this paper, the ideas behind the conditional MSE estimator (8) can be applied here as well. To see this, assume that these areas are numbered last, i.e., if \( D' \) areas have non-zero sample then \( n_i > 0 \) for \( h \leq D' \) and \( n_h = 0 \) for \( h > D' \). For \( i > D' \) the ‘synthetic EBLUP’ for \( m_i \) is

\[
\hat{m}_i^{\text{SYN-EBLUP}} = \mathbf{x}_i^T \hat{\beta}^{\text{EBLUE}} = (w_{is}^{\text{SYN-EBLUP}})^T y_s
\]

\[
= \sum_{h=1}^{D'} \sum_{j \in c_h} w_{ij}^{\text{SYN-EBLUP}} y_j
\]

where

\[
w_{is}^{\text{SYN-EBLUP}} = (w_{ij}^{\text{SYN-EBLUP}})^T = \hat{H}_i^T \mathbf{x}_i.
\]

Clearly (23) is a pseudo-linear estimator, and so we can use (7) to estimate its prediction variance, observing that since \( n_i = 0 \), \( a_j = N_j w_{EJLUP} \) and so (7) becomes

\[
\hat{V}(\hat{m}_i^{\text{SYN-EBLUP}}) = \sum_{j \in c_i} (y_j^{\text{SYN-EBLUP}})^2 + N_i^{-1} n_i^{-1} \hat{\lambda}_{ij}^2 (y_j - \hat{\mu}_j)^2.
\]

Unfortunately, since there is no sample in area \( i \), we cannot use (9) to estimate the area-specific bias (2) of \( \hat{m}_i^{\text{SYN-EBLUP}} \). However, under the linear mixed model (12), this bias has expected value

\[
E(\hat{m}_i^{\text{SYN-EBLUP}} - m_i) = \sum_{h=1}^{D'} \sum_{j \in c_h} w_{ij}^{\text{SYN-EBLUP}} (x_j^T \hat{\beta} + z_j^T u_h) - \mathbf{x}_i^T \hat{\beta} - z_i^T u_s.
\]

The conditional expectation of the square of this expected bias, given the area effects \( u_s = (u_s; h = 1, \ldots, D') \) for the sampled areas, is

\[
E\{E^2(\hat{m}_i^{\text{SYN-EBLUP}} - m_i) \mid X, u_s\} = \left\{ \sum_{h=1}^{D'} \sum_{j \in c_h} w_{ij}^{\text{SYN-EBLUP}} (x_j^T \hat{\beta} + z_j^T u_h) - \mathbf{x}_i^T \hat{\beta} \right\}^2 + z_i^T \Omega z_i,
\]

which immediately suggests that for a non-sampled area \( i \) we estimate the squared bias of the synthetic estimator \( \hat{m}_i^{\text{SYN-EBLUP}} \) by

\[
\hat{B}^2(\hat{m}_i^{\text{SYN-EBLUP}}) = \left\{ \sum_{h=1}^{D'} \sum_{j \in c_h} w_{ij}^{\text{SYN-EBLUP}} (x_j^T \hat{\beta}^{\text{EBLUE}} + z_j^T u_h) - \mathbf{x}_i^T \hat{\beta}^{\text{EBLUE}} \right\}^2 + z_i^T \Omega z_i.
\]

Here \( \hat{u}_s \) is the ‘unshrunken’ estimated effect for sampled area \( h \) see (14). Our proposed MSE estimator for \( \hat{m}_i^{\text{SYN-EBLUP}} \) is then the sum of (24) and (25). Note that, unlike (8), this MSE estimator includes no information from area \( i \), and so is not an estimator of the area-specific MSE of (23). In particular, its validity depends completely on the mixed model (12) holding, and so it is not robust to misspecification of this model.

3. Simulation studies of the proposed MSE estimator

In this section we describe results from five simulation studies that aim at assessing the performance of the approach to conditional MSE estimation described in the previous section. Three of these studies are model-based simulations, with population data generated from the linear mixed model (12). The remaining two are design-based simulations, with population data derived from two real survey datasets where linear SAE is of interest.

Given our focus on bias-robustness, the main performance indicator for an MSE estimator in all five studies is its median relative bias, defined by

\[
\text{RB}(M) = \text{median} \left\{ M_i^{-1} \sum_{k=1}^{K} (\hat{M}_{i,k} - M_i) \right\} \times 100.
\]

Here the subscript \( i \) indexes the small areas and the subscript \( k \) indexes the \( K \) Monte Carlo simulations, with \( \hat{M}_{i,k} \) denoting the simulation \( K \) value of the MSE estimator in area \( i \), and \( M_i \) denotes the actual (i.e., Monte Carlo) MSE in area \( i \). Since we would naturally prefer to use the more stable of two approximately unbiased MSE estimators, we also
measured the stability of an MSE estimator by its median percent relative root mean squared error,

\[ \text{RRMSE}(M) = \text{median} \left( K^{-1} \sum_{i=1}^{K} \left( \frac{\hat{M}_i - M_i}{M_i} \right)^2 \right) \times 100. \]

Although the purpose of this paper is not to compare different methods of SAE, it is useful to relate MSE estimation performance for a particular method of SAE to the actual estimation performance of this method. We therefore provide two measures of the relative performance of the SAE methods that were used in our simulations. These are the median percent relative bias

\[ \text{RB}(m) = \text{median} \left( \frac{\sum_{i=1}^{K} (\hat{m}_{ik} - m_{ik})}{m_{ik}} \right) \times 100 \]

and the median percent relative root mean squared error

\[ \text{RRMSE}(m) = \text{median} \left( K^{-1} \sum_{i=1}^{K} \left( \frac{\hat{m}_{ik} - m_{ik}}{m_{ik}} \right)^2 \right) \times 100 \]

of the estimates \( \hat{m}_{ik} \) generated by an estimation method. Note that \( \bar{m}_i = K^{-1} \sum_{k=1}^{K} m_{ik} \) here.

### 3.1 Model-based simulations

The first model-based simulation study was based on population data generated under the mixed model (12) with Gaussian random effects. It used a population size of \( N = 15,000 \), with \( D = 30 \) small areas. Population sizes in the small areas were uniformly distributed over the interval \([443, 542]\) and kept fixed over simulations. At each simulation, population values for \( Y \) were generated under the random intercepts model

\[ y_{ij} = 500 + 1.5x_{ij} + u_i + \epsilon_{ij}, \]

with \( x_{ij} \) drawn from a chi-squared distribution with 20 degrees of freedom. The area effects \( u_i \) and individual effects \( \epsilon_{ij} \) were independently drawn from \( N(0, \sigma_u^2) \) and \( N(0, \sigma_e^2) \) distributions respectively, with the values of \( \sigma_u \) and \( \sigma_e \) shown in rows SIM1-A and SIM1-B of Table 1. A sample of size \( n = 600 \) was selected from each simulated population, with area sample sizes proportional to the fixed area populations, resulting in a median area sample size of \( n_i = 20 \). Sampling was via stratified random sampling, with the strata defined by the small areas. A total of \( K = 1,000 \) simulations were carried out.

Conditions for the second model-based simulation study were the same as in the first, with the exception that the area level random effects and the individual level random effects were independently drawn from mean corrected chi-square distributions respectively. The corresponding values of the area level and individual level variances are shown in rows SIM2-A and SIM2-B in Table 1. Finally, in the third model-based simulation study conditions were kept the same as in SIM1-A and SIM1-B for areas 1-25, but in areas 26-30 the area effects were independently drawn from a normal distribution with a larger variance. We refer to this as a Mixture in Table 1, with variances for areas 1-25 shown in rows SIM3-A and SIM3-B, and variances for areas 26-30 shown in rows SIM3-A* and SIM3-B*. Our objective in this third simulation was to investigate the behaviour of the different methods of MSE estimation for ‘outlier’ areas, and so we show values relating to areas 1-25 and 26-30 separately in Tables 2 and 4. We also replicated all three scenarios above using a reduced overall sample size of \( n = 150 \) (with median area sample size \( n_i = 5 \)). These additional simulations allowed us to investigate the effect of reduced sample sizes on the performance of the MSE estimators.

### Table 1

Parameter values used in model-based simulations

<table>
<thead>
<tr>
<th>Type</th>
<th>Simulation</th>
<th>( \sigma_u^2 )</th>
<th>( \sigma_e^2 )</th>
<th>( \rho = \sigma_e^2/(\sigma_u^2 + \sigma_e^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>SIM1-A</td>
<td>10.40</td>
<td>94.09</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>SIM1-B</td>
<td>40.32</td>
<td>94.09</td>
<td>0.3</td>
</tr>
<tr>
<td>Chi-square</td>
<td>SIM2-A</td>
<td>2.0</td>
<td>10.0</td>
<td>0.1667</td>
</tr>
<tr>
<td></td>
<td>SIM2-B</td>
<td>4.0</td>
<td>10.0</td>
<td>0.2857</td>
</tr>
<tr>
<td>Mixture (areas 1-25)</td>
<td>SIM3-A</td>
<td>10.40</td>
<td>94.09</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td>SIM3-B</td>
<td>40.32</td>
<td>94.09</td>
<td>0.30</td>
</tr>
<tr>
<td>Mixture (areas 26-30)</td>
<td>SIM3-A*</td>
<td>225.0</td>
<td>94.09</td>
<td>0.7051</td>
</tr>
<tr>
<td></td>
<td>SIM3-B*</td>
<td>225.0</td>
<td>94.09</td>
<td>0.7051</td>
</tr>
</tbody>
</table>
Table 2
Median relative biases $RB(m)$ and median relative root mean squared errors $RRMSE(m)$ of estimators of small area means in model-based simulations

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>SIM1-A</th>
<th>SIM1-B</th>
<th>SIM2-A</th>
<th>SIM2-B</th>
<th>SIM3-A</th>
<th>SIM3-B</th>
<th>SIM3-A*</th>
<th>SIM3-B*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>0.005</td>
<td>0.005</td>
<td>0.000</td>
<td>0.000</td>
<td>0.004</td>
<td>0.004</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>0.005</td>
<td>0.006</td>
<td>0.004</td>
<td>-0.002</td>
<td>0.004</td>
<td>0.005</td>
<td>0.006</td>
<td>0.005</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>0.006</td>
<td>0.006</td>
<td>0.005</td>
<td>-0.008</td>
<td>0.007</td>
<td>0.007</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>0.009</td>
<td>0.008</td>
<td>-0.002</td>
<td>0.002</td>
<td>0.015</td>
<td>-0.013</td>
<td>-0.013</td>
<td>-0.013</td>
</tr>
</tbody>
</table>

$RB(m)$, median $n_i = 20$

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>SIM1-A</th>
<th>SIM1-B</th>
<th>SIM2-A</th>
<th>SIM2-B</th>
<th>SIM3-A</th>
<th>SIM3-B</th>
<th>SIM3-A*</th>
<th>SIM3-B*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>0.40</td>
<td>0.40</td>
<td>0.13</td>
<td>0.13</td>
<td>0.40</td>
<td>0.40</td>
<td>0.41</td>
<td>0.41</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>0.35</td>
<td>0.38</td>
<td>0.12</td>
<td>0.13</td>
<td>0.37</td>
<td>0.38</td>
<td>0.45</td>
<td>0.42</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>0.55</td>
<td>0.55</td>
<td>0.41</td>
<td>0.43</td>
<td>0.56</td>
<td>0.56</td>
<td>0.55</td>
<td>0.55</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>0.41</td>
<td>0.41</td>
<td>0.13</td>
<td>0.13</td>
<td>0.41</td>
<td>0.41</td>
<td>0.36</td>
<td>0.36</td>
</tr>
</tbody>
</table>

$RRMSE(m)$, median $n_i = 20$

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>SIM1-A</th>
<th>SIM1-B</th>
<th>SIM2-A</th>
<th>SIM2-B</th>
<th>SIM3-A</th>
<th>SIM3-B</th>
<th>SIM3-A*</th>
<th>SIM3-B*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>-0.002</td>
<td>-0.003</td>
<td>-0.001</td>
<td>0.002</td>
<td>-0.003</td>
<td>-0.004</td>
<td>0.011</td>
<td>0.011</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>0.001</td>
<td>0.005</td>
<td>-0.002</td>
<td>0.003</td>
<td>0.002</td>
<td>-0.001</td>
<td>0.008</td>
<td>0.011</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>-0.002</td>
<td>-0.002</td>
<td>-0.005</td>
<td>0.004</td>
<td>-0.001</td>
<td>-0.002</td>
<td>-0.002</td>
<td>-0.002</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>0.001</td>
<td>-0.003</td>
<td>-0.003</td>
<td>0.014</td>
<td>0.014</td>
</tr>
</tbody>
</table>

$RB(m)$, median $n_i = 5$

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>SIM1-A</th>
<th>SIM1-B</th>
<th>SIM2-A</th>
<th>SIM2-B</th>
<th>SIM3-A</th>
<th>SIM3-B</th>
<th>SIM3-A*</th>
<th>SIM3-B*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>0.81</td>
<td>0.81</td>
<td>0.26</td>
<td>0.26</td>
<td>0.82</td>
<td>0.82</td>
<td>0.80</td>
<td>0.80</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>0.53</td>
<td>0.69</td>
<td>0.19</td>
<td>0.22</td>
<td>0.61</td>
<td>0.71</td>
<td>1.00</td>
<td>0.87</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>1.13</td>
<td>1.13</td>
<td>0.83</td>
<td>0.83</td>
<td>1.13</td>
<td>1.13</td>
<td>1.13</td>
<td>1.13</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>0.81</td>
<td>0.81</td>
<td>0.26</td>
<td>0.26</td>
<td>0.81</td>
<td>0.81</td>
<td>0.80</td>
<td>0.80</td>
</tr>
</tbody>
</table>

$RRMSE(m)$, median $n_i = 5$

Table 2 shows the median bias $RB(m)$ and median relative root mean squared error $RRMSE(m)$ of the SAE methods investigated in our simulations for the two sample sizes ($n = 600$ and 150). These are the synthetic regression estimator (see Rao 2003, page 136), the EBLUP with weights defined by (13), the MBDE with weights defined by (18) and the M-quantile estimator defined by the weights (22). The differences between the various SAE estimators in Table 2 are essentially as one would expect. Bias is not really an issue (to be expected given the population data follow a linear model in all cases), while for Simulation scenarios 1 and 2 the indirect estimator (EBLUP) is the most efficient in terms of $RRMSE$. The M-quantile estimator is the best performer for SIM3-A* and SIM3-B* with $n_i = 20$ but its difference from the regression synthetic estimator reduces for the scenario with the smaller area-specific sample sizes. Note that in this case the M-quantile weights (22) are based on an outlier-robust estimate of the M-quantile coefficient $\hat{\gamma}_i$ for area $i$, defined by the median (rather than the mean) of the M-quantile coefficients of sampled units in this area. Further, as the sample sizes decrease, the $RRMSE$s of all estimators increase, but their relative performances remain the same. Under normality the EBLUP is better than the M-quantile estimator but the differences between these two estimators become smaller as we move away from normality, with the M-quantile estimator more efficient in the mixture model scenarios.

Table 3 sets out the various MSE estimators investigated in our simulations that are based on the approach proposed in this paper. These are collectively referred to as “conditional” MSE estimators below. In Table 4 we show the performances of MSE estimators for the small area estimators considered in Table 2. Note that in addition to the conditional MSE estimators, we provide results for three other MSE estimators for the EBLUP, with PR0 denoting the estimator suggested by Prasad and Rao (1990), see Rao (2003, section 6.2.6). It is noteworthy that PR0 is not an estimator of the area-specific MSE of the EBLUP, but of its MSE under the mixed linear model (12), i.e., averaged over possible realisations of the area effect. In contrast, the MSE estimators PR1 and PR2 in Table 4 are the area-specific versions of PR0 suggested in Rao (2003, section 6.3.2 expressions 6.3.15 and 6.3.16 respectively). Finally, we note that the MSE estimator of the synthetic regression estimator that we used in our simulations is its variance estimator based on a fixed effects population regression model. We denote it by VReg.

The results set out in Table 4 focus on the median biases $RB(M)$ and median relative root mean squared error $RRMSE(M)$ of the various MSE estimators. Not surprisingly, given that all its underlying assumptions are met, the PR0 estimator and its area-specific alternatives, PR1 and PR2, perform very well in both normal scenarios (SIM1-A and SIM1-B) and both chi-squared scenarios (SIM2-A and SIM2-B), with virtually no bias ($n_i = 20$) or small bias when within area sample sizes are very small. For the MSE estimator of the synthetic regression estimator, on the other hand, we see substantial relative bias under all simulation scenarios.
Table 3
Definitions of conditional MSE estimators for different weighting methods

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>Definition of $\hat{\mu}_j, j \in i$</th>
<th>MSE Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>EBLUP (13)</td>
<td>(14)</td>
<td>(8)</td>
</tr>
<tr>
<td>MBDE (18)</td>
<td>(14)</td>
<td>(8)</td>
</tr>
<tr>
<td>M-quantile (22)</td>
<td>$x_j^T \hat{p}(\tilde{q}_j)$</td>
<td>(7) with $\hat{\lambda}_j = 1$</td>
</tr>
<tr>
<td>Synthetic EBLUP (23)</td>
<td>(14)</td>
<td>(24) + (25)</td>
</tr>
</tbody>
</table>

Table 4
Median relative biases $RB(M)$ and median relative root mean squared errors $RRMSE(M)$ for MSE estimators in model-based simulations

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>MSE Estimator</th>
<th>Simulation</th>
<th>SIM1-A</th>
<th>SIM1-B</th>
<th>SIM2-A</th>
<th>SIM2-B</th>
<th>SIM3-A</th>
<th>SIM3-B</th>
<th>SIM3-A*</th>
<th>SIM3-B*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>VReg</td>
<td>$RB(M)$, median $n_i = 20$</td>
<td>7.59</td>
<td>21.82</td>
<td>11.81</td>
<td>20.78</td>
<td>23.66</td>
<td>34.27</td>
<td>23.97</td>
<td>34.64</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>PR0</td>
<td>-0.83</td>
<td>-0.72</td>
<td>0.56</td>
<td>1.16</td>
<td>3.44</td>
<td>0.71</td>
<td>-15.65</td>
<td>-6.51</td>
<td></td>
</tr>
<tr>
<td></td>
<td>PR1</td>
<td>-0.97</td>
<td>-0.72</td>
<td>0.64</td>
<td>1.08</td>
<td>2.94</td>
<td>0.56</td>
<td>-13.70</td>
<td>-5.81</td>
<td></td>
</tr>
<tr>
<td></td>
<td>PR2</td>
<td>-0.92</td>
<td>-0.72</td>
<td>0.64</td>
<td>1.16</td>
<td>3.20</td>
<td>0.61</td>
<td>-14.65</td>
<td>-6.19</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Conditional</td>
<td>3.89</td>
<td>-0.89</td>
<td>3.06</td>
<td>0.93</td>
<td>-0.05</td>
<td>-0.54</td>
<td>-2.56</td>
<td>-1.59</td>
<td></td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>Conditional</td>
<td>-0.81</td>
<td>-0.80</td>
<td>-0.06</td>
<td>-0.42</td>
<td>-0.75</td>
<td>-0.75</td>
<td>-0.98</td>
<td>-0.98</td>
<td></td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>Conditional</td>
<td>-3.10</td>
<td>-1.66</td>
<td>-0.09</td>
<td>-1.90</td>
<td>-5.04</td>
<td>-3.17</td>
<td>11.26</td>
<td>11.04</td>
<td></td>
</tr>
</tbody>
</table>

| Weighting Method | MSE Estimator | $RRMSE(M)$, median $n_i = 20$ | 5.59  | 19.17  | 10.35  | 19.12  | 20.92  | 30.91  | 22.93   | 33.00   |
| Regression       | VReg          | 18          | 51     | 30     | 53     | 59     | 85     | 60      | 86      |
| EBLUP, (13)      | PR0           | 12          | 7      | 15     | 10     | 11     | 7      | 29      | 14      |
|                  | PR1           | 14          | 7      | 17     | 11     | 10     | 7      | 27      | 13      |
|                  | PR2           | 12          | 7      | 16     | 10     | 11     | 7      | 28      | 13      |
|                  | Conditional  | 62          | 31     | 70     | 49     | 31     | 30     | 42      | 32      |
| MBDE, (18)       | Conditional  | 70          | 70     | 126    | 128    | 71     | 71     | 67      | 67      |
| M-quantile, (22) | Conditional  | 32          | 34     | 49     | 48     | 31     | 32     | 48      | 48      |

| Weighting Method | MSE Estimator | $RB(M)$, median $n_i = 5$ | 5.95  | 19.17  | 10.35  | 19.12  | 20.92  | 30.91  | 22.93   | 33.00   |
| Regression       | VReg          | 17          | 46     | 33     | 51     | 54     | 78     | 59      | 83      |
| EBLUP, (13)      | PR0           | 31          | 14     | 33     | 22     | 36     | 16     | 53      | 31      |
|                  | PR1           | 48          | 18     | 44     | 28     | 34     | 16     | 48      | 29      |
|                  | PR2           | 36          | 15     | 36     | 24     | 34     | 15     | 50      | 29      |
|                  | Conditional  | 234         | 81     | 193    | 121    | 86     | 66     | 86      | 70      |
| MBDE, (18)       | Conditional  | 79          | 79     | 133    | 129    | 79     | 79     | 83      | 83      |
| M-quantile, (22) | Conditional  | 62          | 63     | 90     | 97     | 63     | 63     | 122     | 102     |

The conditional MSE estimator for the EBLUP shows positive bias under both the normal (SIM1A) and chi-squared (SIM2A) scenarios, particularly for moderate intra-cluster correlation (3.89% and 37.52% for the normal scenario with 20 and 5 units in each area respectively and 3.06% and 24.11% for the chi-squared scenario with 20 and 5 units in each area respectively). This bias increases with decreasing sample size. However, things change when we examine the results for the outlier components of the mixture model scenarios (SIM3-A* and SIM3-B*). Here we see a substantial negative bias for all three versions of PR (ranging from -30.64% to -5.81% depending on the area sample sizes). In comparison, the conditional MSE estimator for the EBLUP now shows a smaller negative bias (-2.56% and -0.66%) while the same MSE estimator applied to the M-quantile estimator shows an upward bias. The
conditional MSE estimator for the MBDE is essentially unbiased. Given that as far as MSE estimation is concerned, positive bias is preferable to negative bias, it seems clear that the proposed conditional MSE estimator is better able to handle this outlier situation. Figure 1 graphically illustrates this point for sample size \( n = 600 \). Here we show the area-specific RMSEs and the average (over the simulations) of the estimated RMSEs in each of the 30 areas for the mixture simulations SIM3-A and SIM3-A*, with the vertical line delineating the five ‘outlier’ areas. In the top panel of this plot we see that the PR0 estimator is unable to detect the step increase in the MSE of the EBLUP for these ‘outlier’ areas, being biased slightly high in the ‘well-behaved’ areas and then biased rather low in the ‘outlier’ areas. In contrast, the conditional MSE estimator for the EBLUP and the MBDE tracks the area specific RMSEs rather well, while the same MSE estimator based on M-quantile weights tends to be biased low in the ‘well-behaved’ areas, and biased high in the ‘outlier’ areas, which can be argued as being perhaps a rather better outcome than that recorded by the PR0 estimator in this simulation. It should be noted here that in certain circumstances an assumed model can be revised after outlier detection. However, this requires a sufficiently large number of detected outliers to permit their separate modelling. This is unlikely to happen in practice. Also, particular care must be taken with extrapolation of these results to the case of very small area sample sizes because of the instability that the conditional MSE estimator can exhibit in this case.

Table 4 also shows the relative RMSEs of the different MSE estimators across the three types of model-based simulation. Here we see that all three versions of the PR estimator of the MSE of the EBLUP are more stable than the conditional MSE estimator of the EBLUP (12% for PR vs. 62% for the conditional MSE for SIM1-A with \( n_i = 20 \) and 31% for PR vs. 234% for the conditional MSE for SIM1-A with \( n_i = 5 \)). These differences decrease under scenarios SIM3-A* and SIM3-B*, however, although the PR MSE estimator remains more stable (13% for PR vs. 32% for the conditional MSE estimator for SIM3-B* with \( n_i = 20 \) and 29% for the PR MSE estimator vs. 70% for the conditional MSE estimator for SIM3-B* with \( n_i = 5 \)). The same is true for the conditional MSE estimators of the MBDE and the M-quantile estimators. Essentially, given sample data that follow a mixed linear model, the PR MSE estimator of MSE is very stable, while the conditional MSE estimator is more variable.

In summary, although all methods of MSE estimation that we evaluated exhibited some bias for very small area sample sizes, our model-based simulation results provide evidence that for larger area sample sizes the conditional MSE estimation method (8) is bias robust when applied to the three pseudo-linear small area estimators EBLUP, MBDE and M-quantile. For very small area sample sizes its bias robustness is less evident. As one might expect, the model dependent ‘area-averaged’ MSE estimator PR0 for the EBLUP exhibits bias under model failure. The fact that we observed rather similar behaviour for the area-specific versions PR1 and PR2 of this MSE estimator indicates that ‘area specific’ does not necessarily mean ‘bias robust’. In particular, the fact that PR1 and PR2 behave very similarly to PR0 may be because the area-specific components of PR1 and PR2 are of lower order and all three MSE estimators have the same leading term, which is not area-specific. Our results also show that the conditional MSE estimator (8) is much more variable than the model dependent PR MSE estimator, even for moderate area sample sizes.

3.2 Design-based simulations

What happens when, as in real life, we cannot be confident that our data follow a linear mixed model? In order to investigate this situation, we report results from two design-based simulation studies, both based on realistic populations, where a linear model assumption is essentially an approximation. The first involved a sample of 3,591 households spread across \( D = 36 \) districts of Albania that participated in the 2002 Albanian Living Standards Measurement Study. This sample was bootstrapped to create a realistic population of \( N = 724,782 \) households by re-sampling with replacement with probability proportional to a household’s sample weight. A total of \( K = 1,000 \) independent stratified random samples were then drawn from this bootstrap population, with total sample size equal to that of the original sample and with districts defining the strata. Sample sizes within districts were the same as in the original sample, and varied between 8 and 688 (with median district sample size equal to 56). The \( Y \) variable of interest was household per capita consumption expenditure (HCE) and \( X \) was defined by three zero-one variables (ownership of television, parabolic antenna and land). The aim was to estimate the average value of HCE for each district. In the original 2002 survey, the linear relationship between HCE and the three variables making up \( X \) was rather weak, with very low predictive power. In particular, only ownership of land was significantly related to HCE at the five percent level. This fit was considerably improved by extending the linear model to include random intercepts, defined by independent district effects. These explained approximately 10 per cent of the residual variation in this model.

Statistics Canada, Catalogue No. 12-001-X
Figure 1 Area specific values of true RMSE (solid line) and average estimated RMSE (dashed line) obtained in the mixture-based simulations SIM3-A and SIM3-A*. Values for the PR0 estimator are indicated by $\Delta$ while those for the conditional estimator are indicated by $\varphi$. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators. Vertical line separates areas 26-30 with ‘outlier’ effects from ‘well-behaved’ areas 1-25. Total sample size is 600 with area-specific sample sizes equal to 20.
The second design-based simulation study was based on an ‘outlier free’ version of the population of Australian broadacre farms that was used in the simulation studies reported in Chambers and Tzavidis (2006) and Chandra and Chambers (2009). In particular, this population was defined by bootstrapping a sub-sample of 1,579 ‘non-outlier’ farms that participated in the Australian Agricultural and Grazing Industries Survey (AAGIS) to create a population of \( N = 78,072 \) farms by re-sampling from the original AAGIS sample with probability proportional to a farm’s sample weight. The small areas of interest in this case were the \( D = 28 \) broadacre farming regions represented in this subsample. The design-based simulation was carried out by selecting \( K = 1,000 \) independent stratified random samples from this bootstrap population, with strata defined by the regions and with stratum sample sizes defined by those in the original AAGIS sample. These sample sizes vary from 6 to 117, with a median region sample size of 53. Here \( Y \) is Total Cash Costs (TCC) associated with operation of the farm, and \( X \) is a vector that includes farm area (Area), effects for six post-strata defined by three climatic zones and two farm size bands as well as the interactions of these variables. In the original AAGIS sample the relationship between TCC and Area varies significantly between the six post-strata, with an overall Rsquared value of approximately 0.46 after the deletion of two outliers. The fixed effects in the prediction model were therefore specified as corresponding to a separate linear fit of TCC in terms of Area in each post-stratum. Random effects (necessary for computation of the EBLUP and the MBDE, but not the M-quantile predictor) were defined as independent regional effects (i.e., a random intercepts specification) on the basis that in the original AAGIS sample the between region variance component explains about 3 per cent of the total residual variability with the two outliers removed. The aim was to estimate the regional averages of TCC.

Tables 5 and 6 show the median relative biases and the median relative RMSEs of different estimators and corresponding estimators of the MSEs of these estimators based on the \( K = 1,000 \) independent stratified samples taken from the Albanian and AAGIS populations respectively. It is noteworthy that in spite of the fact that the linear mixed models fitted to both the Albanian and AAGIS data appear reasonable, the gains from adoption of SAE methods based on them do not lead to substantial improvements in efficiency given the original regional sample sizes for these surveys. On the other hand, the M-quantile estimator, which is not based on a random effects specification, works well both in terms of bias and MSE for the AAGIS population in this case (Table 6, Median \( \hat{n}_i = 53 \)), while the EBLUP, although the best performer in terms of MSE for the Albanian population (Table 5, Median \( \hat{n}_i = 56 \)), also records the highest biases (albeit still small, with the largest less than 2%) for both populations given the original area sample sizes. The survey regression estimator performs well, although for both populations there are indirect estimators that perform somewhat better. Design-based simulations based on the Albanian and AAGIS populations were also carried out using smaller area sample sizes than in the original surveys. In particular, the overall sample size was reduced for the Albanian population to \( n = 291 \) (with a median district sample size of 9). Similarly, the overall sample size was reduced for the AAGIS population to \( n = 233 \) (with a median regional sample size of 8). As expected the RMSE of the point estimators increases as the area sample sizes decrease. Overall, the EBLUP improves its RMSE performance relative to all other estimators given these smaller sample sizes. However, since the realism of these reduced sample size designs is somewhat questionable, we do not place too much emphasis on results derived from them, noting only that they are useful for assessing the performance of MSE estimators with realistic data and with very small sample sizes.

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>Median ( \hat{n}_i = 56 )</th>
<th>Median ( \hat{n}_i = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimator</strong></td>
<td>RB(( m ))</td>
<td>RRMSE(( m ))</td>
</tr>
<tr>
<td>Regression</td>
<td>0.04</td>
<td>6.25</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>0.42</td>
<td>5.90</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>0.03</td>
<td>6.14</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>0.04</td>
<td>6.07</td>
</tr>
<tr>
<td><strong>Method/MSE</strong></td>
<td>RB(( m ))</td>
<td>RRMSE(( m ))</td>
</tr>
<tr>
<td>Regression /VReg</td>
<td>17.6</td>
<td>42</td>
</tr>
<tr>
<td>EBLUP/PR0</td>
<td>14.6</td>
<td>44</td>
</tr>
<tr>
<td>EBLUP/PR1</td>
<td>14.4</td>
<td>43</td>
</tr>
<tr>
<td>EBLUP/PR2</td>
<td>14.5</td>
<td>43</td>
</tr>
<tr>
<td>EBLUP/Conditional</td>
<td>0.1</td>
<td>24</td>
</tr>
<tr>
<td>MBDE/Conditional</td>
<td>-0.8</td>
<td>25</td>
</tr>
<tr>
<td>M-quantile/Conditional</td>
<td>2.9</td>
<td>27</td>
</tr>
</tbody>
</table>

Statistics Canada, Catalogue No. 12-001-X
Table 6
Performances of estimators of regional means and their MSE estimators – AAGIS farm population

<table>
<thead>
<tr>
<th>Weighting Method</th>
<th>Median $n_j = 53$</th>
<th>Median $n_j = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{RB}(m)$</td>
<td>$\text{RRMSE}(m)$</td>
</tr>
<tr>
<td>Regression</td>
<td>0.03</td>
<td>13.36</td>
</tr>
<tr>
<td>EBLUP, (13)</td>
<td>1.64</td>
<td>13.53</td>
</tr>
<tr>
<td>MBDE, (18)</td>
<td>-0.73</td>
<td>14.26</td>
</tr>
<tr>
<td>M-quantile, (22)</td>
<td>-0.04</td>
<td>11.68</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method/MSE</th>
<th>RB($M$)</th>
<th>RRMSE($M$)</th>
<th>RB($M$)</th>
<th>RRMSE($M$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression /VReg</td>
<td>74.1</td>
<td>406</td>
<td>54.7</td>
<td>867</td>
</tr>
<tr>
<td>EBLUP/PR0</td>
<td>22.4</td>
<td>131</td>
<td>17.7</td>
<td>374</td>
</tr>
<tr>
<td>EBLUP/PR1</td>
<td>19.5</td>
<td>137</td>
<td>19.0</td>
<td>367</td>
</tr>
<tr>
<td>EBLUP/PR2</td>
<td>21.0</td>
<td>123</td>
<td>31.1</td>
<td>444</td>
</tr>
<tr>
<td>EBLUP/Conditional</td>
<td>5.5</td>
<td>132</td>
<td>17.8</td>
<td>255</td>
</tr>
<tr>
<td>MBDE/Conditional</td>
<td>-0.5</td>
<td>181</td>
<td>0.9</td>
<td>318</td>
</tr>
<tr>
<td>M-quantile/Conditional</td>
<td>-0.7</td>
<td>69</td>
<td>-1.9</td>
<td>212</td>
</tr>
</tbody>
</table>

Focusing on the simulation results obtained using the original regional sample sizes, we see that all three PR-based MSE estimators for the EBLUP display a substantial upward bias in both sets of design-based simulations as well as larger (Albanian population, Table 5) or comparable (AAGIS population, Table 6) instability to the conditional MSE estimators. For the Albanian population all three versions of the conditional MSE estimator are essentially unbiased whereas for the AAGIS population all three versions of the conditional MSE estimator display small or moderate bias.

It is noteworthy that for the Albanian population (Table 5) the relative performances of the PR MSE estimators improve with smaller samples. However, this is because the conditional MSE estimators then become more unstable. For these very small area samples the conditional MSE estimator is less biased than the PR MSE estimator (7.7% vs 10.5%) but is also more unstable (RRMSE of conditional MSE estimator is 99% vs 50% for the PR MSE estimator). This is, however, not the case for the AAGIS population with median $n_j = 8$. In this case, the PR-based MSE estimators perform badly, with the conditional MSE estimators being both less biased and more stable.

The MSE estimator of the regression estimator exhibits moderate or high bias for both populations and all simulation scenarios. For the Albanian population it appears to be competitive to the other MSE estimators in terms of RRMSE but for the AAGIS population it is clearly less stable than the other MSE estimators. Finally, the conditional MSE estimator of the M-quantile estimator performs well with small relative bias and good stability for all simulation scenarios and both populations with the exception of the Albanian population with median $n_j = 9$ where its RRMSE is 75%.

An insight into the reasons for these differences in behaviour can be obtained by examining the area specific RMSE values displayed in Figure 2 for the Albanian population and in Figure 3 for the AAGIS population. Note that in both cases the sample sizes are those from the original surveys. Thus, in Figure 2 we see that all three conditional MSE estimators track the district-specific design-based RMSEs of their respective estimators exceptionally well. In contrast, the PR0 estimator does not seem to be able to capture between district differences in the design-based RMSE of the EBLUP. In Figure 3 we see that the conditional estimator of the MSE of the M-quantile estimator performs extremely well in all regions, with the corresponding estimator of the MSE of the MBDE also performing well in all regions except one (region 6) where it substantially overestimates the design-based RMSE of this predictor. This region is noteworthy because samples that are unbalanced with respect to Area within the region lead to negative weights under the assumed linear mixed model.

The picture becomes more complex when one considers the region-specific RMSE estimation performance of the EBLUP in Figure 3. Here we see that the conditional estimator of the MSE of the EBLUP clearly tracks the region-specific design-based RMSE of this predictor better than the PR0 MSE estimator. With the exception of region 6 (where sample balance is a problem), there seems to be little regional variation in the value of the PR0 estimator of the RMSE of the EBLUP, indicating a serious bias problem.

As noted earlier, it is not uncommon to want to produce an estimate for a small area where there is no sample. In such cases, one has to rely completely on the correctness of the model specification. In Table 7 we illustrate the importance of this assumption by contrasting the estimation and MSE estimation performances of the EBLUP for sampled areas with that of the Synthetic EBLUP for areas where no sample data are available. Two situations are shown. The first is a modification of the model-based SIM1-A simulation with a small average sample size and with five zero sample areas. The second is a similar small sample modification of the design-based simulation based on the
AAGIS population, with four zero sample areas. It is clear that when the model underpinning the EBLUP actually holds (i.e., SIM1-A), estimation and MSE estimation (either based on PR0, or on the conditional alternative) works well. The problem is that when there is some doubt about how well this model holds (as in the AAGIS population), then the EBLUP can fail, and our estimator of its MSE can also fail to identify this problem. This is nicely illustrated by the results for the AAGIS population in Table 7 where we see that both the PR0 and conditional MSE estimators for the Synthetic EBLUP completely fail to identify the large positive bias of the Synthetic EBLUP and so end up with a large downward bias.

Figure 2  District level values of true design-based RMSE (solid line) and average estimated RMSE (dashed line) obtained in the design-based simulations using the Albanian household population. Districts are ordered in terms of increasing population size. Values for the PR0 estimator are indicated by Δ while those for the conditional estimator are indicated by φ. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators.
Figure 3  
Regional values of true design-based RMSE (solid line) and average estimated RMSE (dashed line) obtained in the design-based simulations using the AAGIS farm population. Regions are ordered in terms of increasing population size. Values for the PRO estimator are indicated by $\Delta$ while those for the conditional estimator are indicated by $\circ$. Plots show results for the EBLUP (top), MBDE (centre) and M-quantile (bottom) estimators.

Table 7  
Performance of EBLUP and MSE estimators when there are areas with zero sample

<table>
<thead>
<tr>
<th>Weighting Method/ Estimator</th>
<th>SIM1-A, median $n_I = 10$</th>
<th>AAGIS, median $n_I = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RB($m$)</td>
<td>RRMSE($m$)</td>
</tr>
<tr>
<td>Areas with $n_I &gt; 0$</td>
<td>(13)/EBLUP</td>
<td>0.00</td>
</tr>
<tr>
<td>Areas with $n_I = 0$</td>
<td>(23)/Synthetic EBLUP</td>
<td>-0.05</td>
</tr>
<tr>
<td>MSE Estimator</td>
<td>RB($M$)</td>
<td>RRMSE($M$)</td>
</tr>
<tr>
<td>Areas with $n_I &gt; 0$</td>
<td>(13)/PRO</td>
<td>0.5</td>
</tr>
<tr>
<td>Areas with $n_I = 0$</td>
<td>(23)/Conditional</td>
<td>0.7</td>
</tr>
<tr>
<td>Areas with $n_I = 0$</td>
<td>(23)/Conditional</td>
<td>-1.8</td>
</tr>
<tr>
<td></td>
<td>(23)/Conditional</td>
<td>-3.6</td>
</tr>
</tbody>
</table>
4. Conclusions and discussion

In this paper we propose a bias-robust and easily implemented method of estimating the conditional MSE of pseudo-linear estimators of small area means (and totals). Our empirical results show that this method of MSE estimation performs reasonably well in terms of bias when used to estimate the model-based MSE and the design-based MSE of the three rather different pseudo-linear estimators considered in this paper. However, this improved bias performance comes at the cost of increased variability. In particular, when area sample sizes are very small, we do not recommend use of our proposed method of MSE estimation for a conditionally biased estimator like the EBLUP.

The EBLUP is widely used in SAE, and in this context the model-dependent MSE estimator PR0 for the EBLUP suggested by Prasad and Rao (1990) is unbiased when its model assumptions are valid (SIM1-A/B and SIM2-A/B in our model-based simulations) but is biased in the presence of outlier area effects (SIM3-A/A* and SIM3-B/B*). It was also the most stable MSE estimator in the model-based simulations. However, its area-averaged construction meant that it did not track the area-specific MSE of the EBLUP in both our design-based simulations, where the correctness of the assumed linear mixed model could only be considered as approximate. This suggests that our proposed conditional MSE estimation method should be considered as an alternative to PR0 in situations where there is some doubt about the correctness of the specification of the small area linear mixed model or where the area sample sizes are not small. Some idea of what constitutes a small sample size can be deduced from the empirical results presented in this paper.

If there is doubt about the validity of the assumed linear mixed model, the user could consider estimation based on a more widely applicable alternative model, e.g., the M-quantile model, or replace the EBLUP by a more outlier-robust alternative (Sinha and Rao 2009). In the former case the approach that we propose in this paper is currently the only analytical approach to MSE estimation, while in the latter case it provides an analytic alternative to more computationally intensive bootstrap methods of MSE estimation. Note however, that for very small area-specific sample sizes the bias-robust MSE estimator proposed in this paper remains unstable.

A future line of research could be to compare the analytic MSE estimation method proposed in this paper with bootstrap-based MSE estimators, e.g., the nonparametric bootstrap MSE estimator of the M-quantile estimator proposed by Tzavidis, Marchetti and Chambers (2010), and the bootstrap MSE estimator for the Robust EBLUP estimator proposed by Sinha and Rao (2009). A key issue in this investigation will be to investigate whether alternative bootstrap MSE estimators are more stable, especially for small area-specific sample sizes.

The extension of the conditional MSE approach to non-linear SAE situations remains to be done. However, since this approach is closely linked to robust population level MSE estimation based on Taylor series linearisation (as well as jackknife estimation of MSE, see Valliant, Dorfman and Royall 2000, section 5.4.2), it should be possible to develop appropriate extensions for corresponding small area non-linear estimation methods. Although the relevant results are not provided here, some evidence for this is that the conditional MSE estimation method described in this paper has already been used to estimate the MSE of the MBDE when it is applied to variables that do not lend themselves to linear mixed modelling, e.g., those with a high proportion of zero values (Chandra and Chambers 2009), and categorical variables (Chandra, Chambers and Salvati 2011). More recently, the approach has also been used to estimate the MSE of geographically weighted M-quantile small area estimators in situations where the small area values are spatially correlated (Salvati, Tzavidis, Pratesi and Chambers 2011). It has also been used by Salvati, Chandra, Ranalli and Chambers (2010) to estimate the MSE of small area estimators based on a nonparametric small area model (Opsomer, Claeskens, Ranalli, Kauermann and Breidt 2008).

As is clear from the development in this paper, our preferred approach to MSE estimation assumes that the MSE of real interest is that defined by the area-specific model (1). This is in contrast to the usual approach to defining MSE in SAE, which adopts an area-averaged MSE concept as the appropriate measure of the accuracy of a small area estimator. As pointed out by Longford (2007), the ultimate aim in SAE is to make inferences about small area characteristics conditional on the realised (but unknown) values of small area effects, i.e., with respect to (1). One can consider this to be a design-based objective (as in Longford 2007), or, as we prefer, a model-based objective that does not quite fit into the usual random effects framework for SAE. In either case we are interested in variability that is with respect to fixed area-specific expected values. This is consistent with the concept of variability that is typically applied in population level inference.

Acknowledgements

The authors would like to acknowledge the valuable comments and suggestions of the Editor, the Associate Editor and two referees. These led to a considerable improvement in the paper.
References


