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Variance Estimation for the Current Employment Survey

Jun Shao and Shail Butani

Abstract

Like most other surveys, nonresponse often occurs in the Current Employment Survey conducted monthly by the U.S. Bureau of Labor Statistics (BLS). In given month, imputation using reported data from previous months generally provides more efficient survey estimators than ignoring nonrespondents and adjusting survey weights. However, imputation also has an effect on variance estimation: treating imputed values as reported data and applying a standard variance estimation method leads to negatively biased variance estimators. In this article we propose some variance estimators using the grouped balanced half sample method and re-imputation to take imputation into account. Some simulation results for the finite sample performance of the imputed survey estimators and their variance estimators are presented.

Key Words: Balanced half samples; Non-negligible sampling fractions; Ratio imputation; Stratified sampling.

1. Introduction

The Current Employment Survey (CES), commonly known as the payroll survey, is conducted monthly by the U.S. Bureau of Labor Statistics (BLS). The data are obtained from establishments on a monthly basis by various automated methods including computer assisted telephone interviews, touchtone data entry, FAX, electronic data interchange, mail, etc. The main variables are the employment, production or non-supervisory workers and their working hours and earnings on nonagricultural establishments payrolls. Population employment counts are obtained once a year from Unemployment Insurance administrative records.

Nonresponse often occurs in the CES. In any particular month, imputation using reported data from previous months generally provides more efficient survey estimators than using reported data in the current month only and adjusting survey weights. This is particularly true in the CES because the nonresponse rate is about 60% and about 60% of the nonrespondents in a given month may become available one or several months later so that these data can be used as “reported data from previous months” (historical data) in a future month.

However, it is well known that treating imputed values as reported data and applying a standard variance estimation method leads to biased (often negatively biased) variance estimators. Valid variance estimators can be derived under some assumptions on sampling designs, imputation methods, and response mechanisms (and, sometimes, models that generate data).

The purpose of this article is to study variance estimation for the CES. After describing the sampling design and the imputation procedure currently used for the CES in section 2, we derive valid (asymptotically unbiased and consistent) variance estimators for imputed survey estimators in section 3. To simplify the computation of variation estimators, we propose some approximations in section 4 and study their properties by simulation in section 5. Some conclusions are made in section 6. Although our study is motivated by the CES, we believe that our results are general and applicable to any survey that adopts a similar sampling design and a similar imputation method.

2. Sampling Design and Imputation

The CES adopts the following stratified probability sampling design. Let \( P \) be a finite population containing a set of establishments \( \{1, \ldots, N\} \), which is stratified by the type of industry and by the size of the establishment. Within the \( h \)th stratum, a sample of size \( n_h \geq 2 \) is taken without replacement from \( N_h \) population units, using probability sampling independently across strata. The sampling fractions \( n_h / N_h \) are not necessarily negligible; for some strata with large establishment sizes, \( n_h = N_h \). Let \( S \) denote the sample. For \( i \in S \), at month \( t = 0, 1, \ldots, T \), values on the number of employees \( (y_{t}^{E}) \), the number of non-supervisory workers \( (y_{t}^{W}) \), the number of hours worked \( (y_{t}^{H}) \), and the weekly pay \( (y_{t}^{P}) \) are observed (if there is no nonresponse). Let \( y_{t,i} \) denote any of \( y_{t}^{E}, y_{t}^{W}, y_{t}^{H}, \) or \( y_{t}^{P} \). In CES, the main parameters of interest are population totals \( Y_{t} = \sum_{i \in P} y_{t,i} \), \( t = 1, \ldots, T \). Since population totals can be obtained once a year from administrative records, we assume without loss of generality that \( Y_{0} \) is known. If there is no nonresponse, \( Y_{t} \) is estimated by a ratio estimator

\[
\hat{Y}_{t} = Y_{0} \sum_{i \in S} w_{i} y_{t,i} / \sum_{i \in S} w_{i} y_{0,i}, \quad t = 1, \ldots, T,
\]

where \( w_{i} \) is the survey weight for the \( i \)th unit in the sample and the \( h \)th stratum.

In our research, starting from month 1, nonrespondents are imputed using the imputation method proposed in Butani, Harter and Wolter (1997), as described below. Imputation is carried out within an imputation cell, which is the same as stratum or a union of strata. Imputed values in months 1, ..., \( t - 1 \) are carried over to impute nonrespondents in month \( t \), unless nonrespondents in months 1, ..., \( t - 1 \) become respondents prior to month \( t \).

1. The number of employees. If \( y^{E}_{t,i} \) is a nonrespondent, it is imputed by

\[
\hat{y}^{E}_{t,i} = \hat{\alpha}_t \bar{y}^{E}_{t-1,i},
\]

where \( \bar{y}^{E}_{t-1,i} = y^{E}_{t-1,i} \) (reported value) if \( y^{E}_{t-1,i} \) is available at month \( t \) and otherwise \( \bar{y}^{E}_{t-1,i} \) is an imputed value,

\[
\hat{\alpha}_t = \frac{\sum_{j \in R_k} w_j y^{E}_{t,j}}{\sum_{j \in R_k} w_j y^{E}_{t-1,j}},
\]

and \( R_k \) is the set of all reporting units for months \( t \) and \( t - 1 \).

2. The number of non-supervisory workers. If \( y^{W}_{t,i} \) is a nonrespondent, it is imputed by

\[
\hat{y}^{W}_{t,i} = \hat{\gamma}_t \bar{y}^{W}_{t-1,i} \bar{y}^{E}_{t-1,i} \bar{y}^{E}_{t-1,i},
\]

where \( \bar{y}^{W}_{t-1,i} \) is defined similarly to \( \bar{y}^{E}_{t-1,i} \).

3. The number of hours worked. If \( y^{H}_{t,i} \) is a nonrespondent, it is imputed by

\[
\hat{y}^{H}_{t,i} = \hat{\gamma}_t \bar{y}^{H}_{t-1,i} \bar{y}^{W}_{t-1,i} / \bar{y}^{E}_{t-1,i},
\]

where \( \bar{y}^{H}_{t-1,i} \) is defined similarly to \( \bar{y}^{E}_{t-1,i} \) and

\[
\hat{\gamma}_t = \frac{\sum_{j \in R_k} w_j y^{H}_{t,j} / \sum_{j \in R_k} w_j y^{W}_{t,j} \sum_{j \in R_k} w_j y^{E}_{t-1,j}}{\sum_{j \in R_k} w_j y^{E}_{t-1,j} / \sum_{j \in R_k} w_j y^{W}_{t-1,j}}.
\]

4. The weekly pay. If \( y^{P}_{t,i} \) is a nonrespondent, it is imputed by

\[
\hat{y}^{P}_{t,i} = \hat{\beta}_t \bar{y}^{P}_{t-1,i} \bar{y}^{H}_{t-1,i} \bar{y}^{E}_{t-1,i},
\]

where \( \bar{y}^{P}_{t-1,i} \) is defined similarly to \( \bar{y}^{E}_{t-1,i} \) and

\[
\hat{\beta}_t = \frac{\sum_{j \in R_k} w_j y^{P}_{t,j} / \sum_{j \in R_k} w_j y^{H}_{t,j} \sum_{j \in R_k} w_j y^{E}_{t-1,j}}{\sum_{j \in R_k} w_j y^{E}_{t-1,j} / \sum_{j \in R_k} w_j y^{H}_{t-1,j}}.
\]

Once nonrespondents are imputed, estimated monthly totals are calculated according to (1) by treating imputed values as reported data.

Assume that the population \( P \) is divided into \( K \) disjoint imputation cells \( P_1, ..., P_K \) and for each \( k \),

\[
y^{E}_{t,i} = \alpha_{t,k} y^{E}_{t-1,i} + \sqrt{\gamma^{E}_{t-1,i} \epsilon^{E}_{t,i}},
\]

\[
E_m(y^{E}_{t,i}) = \mu_{t,k}, \quad E_m(\epsilon^{E}_{t,i}) = 0, \quad i \in P_k, t = 1, 2, ...
\]

\[
E_m(y^{E}_{t,i}) = \sqrt{\gamma^{E}_{t-1,i} \sigma^2_{E}},
\]

where \( y^{E}_{t,i} \) denotes any of \( y^{E}_{t,i}, y^{W}_{t,i}, y^{H}_{t,i}, \) or \( y^{P}_{t,i} \) and \( E_m \) and \( V_m \) are the model (marginal) expectation and variance, respectively, \( \alpha_{t,k} \) and \( \sigma^2_{E} \) are unknown parameters, \( \epsilon^{E}_{t,i} \)'s are iid and the two processes \( \{y^{E}_{t,i}\} \) and \( \{\epsilon^{E}_{t,i}\} \) are independent. Within each \( P_k \), it is assumed that the response indicator \( a_{t,k} \) \( (= 1 \text{ if } y^{E}_{t,i} \text{ is a respondent and } = 0 \text{ otherwise}) \) and \( y^{E}_{t,i} \) are independent, given \( y^{E}_{t-1,i}, a_{t-1,k} \), \( s = 1, 2, ..., t \). Under this response mechanism, which is called unconfounded response mechanism (Lee, Rancourt and Särndal 1994), \( a_{t,k} \) and \( y^{E}_{t,i} \) are dependent, but through \( y^{E}_{t-k,i}, a_{t-k,k} \), \( s = 1, 2, ..., t \). It is more general than the assumption that \( \{y^{E}_{t,i}, ..., y^{E}_{t-1,i}\} \) and \( \{a_{t-1,i}, ..., a_{t+k-1,i}\} \) are independent. Finally, response indicators from different units are assumed to be independent. Under these assumptions, the estimators \( \hat{\gamma} \) based on imputed data as described in the previous section are asymptotically unbiased with respect to the joint expectation under model (2) and sampling from the finite population.

In the CES, the imputation cells are unions of strata so that

\[
\sum_{i \in S \cap P_k} w_i = M_k, \quad k = 1, ..., K,
\]

where \( M_k \) is the number of population units in the \( k \text{th} \) imputation cell \( P_k \). Consequently, the \( \hat{\gamma} \) are conditionally unbiased with respect to the model expectation (given \( S \)), i.e.,

\[
E_m(\hat{\gamma} - \gamma) = 0.
\]

### 3. Variance Estimation

Let \( E_s \) and \( V_s \) be the sampling expectation and variance, respectively, and \( V \) be the overall variance. Then

\[
V(\hat{\gamma} - \gamma) = E_s[V_m(\hat{\gamma} - \gamma)] + V_s[E_m(\hat{\gamma} - \gamma)]
\]

\[
= E_s[V_m(\hat{\gamma} - \gamma)],
\]

since \( E_m(\hat{\gamma} - \gamma) = 0 \). Furthermore, it is shown in the Appendix that

\[
V_m(\hat{\gamma} - \gamma) = V_m(\hat{\gamma}) - V_m(\gamma).
\]
Note that (4) is obvious in the case of no nonresponse.

Because of (3) the estimation of \( V(Y_t - Y_t) \) is the same as the estimation of \( V_m(Y_t - Y_t) \). Also, because of (4), we can first derive estimators \( v_{i1} \) and \( v_{i2} \) of \( V_m(Y_t) \) and \( V_m(Y_t) \), respectively, and then take the difference \( v_{i1} - v_{i2} \) as our variance estimator for \( Y_t \). Since \( V_m(Y_t) \) is a conditional variance, given \( S \), we do not need to consider the sampling fractions \( n_h / N_h \) in the estimation of \( V_m(Y_t) \).

We first consider the estimation of \( V_m(Y_t) \). If an approximate formula of \( V_m(Y_t) \) can be derived, then we can directly estimate \( V_m(Y_t) \) by substitution. The explicit form of \( Y_t \) however, is very complex when \( t \) is not small so that the derivation of \( V_m(Y_t) \) is very difficult. Thus, in the CES we adopt a grouped half sample method that incorporates Rao and Shao’s (1992) adjustment (or re-imputation) to take imputation into account. Specifically, sampled units in each stratum are randomly grouped into two groups. \( R \) half samples are created using a Hadamard matrix, where \( H + 1 \leq R \leq H + 4 \) and \( H \) is the number of strata. For the \( r \)th half sample and the \( i \)th sampled unit, define

\[
    w_i^{(r)} = \begin{cases} 
    (1 + 0.5)w_i & \text{if the unit is in the } r \text{th half sample} \\
    (1 - 0.5)w_i & \text{if the unit is not in the } r \text{th half sample},
    \end{cases}
\]

where \( w_i \) is the original survey weight. Let \( \tilde{Y}_t^{(r)} \) be the same as \( Y_t \) except that the weights \( w_i \) are replaced by the \( w_i^{(r)} \), including the weights used in imputation (i.e., \( \hat{\alpha}_i, \hat{\beta}_i \), and \( \hat{\beta}_i \) are re-computed for every \( r \), which is equivalent to Rao and Shao’s adjustment). A grouped half sample variance estimator of \( V_m(Y_t) \) is

\[
    v_{i1} = 4 \sum_{r=1}^R \left( \tilde{Y}_t^{(r)} - \frac{1}{R} \sum_{r=1}^R \tilde{Y}_t^{(r)} \right)^2.
\]

Note that the use of 0.5, instead of 1, in the construction of \( w_i^{(r)} \) is based on Fay’s method (Dippo, Fay and Morganstein 1984; Judkins 1990; Rao and Shao 1999). Asymptotically, \( v_{i1} \) is unbiased and consistent for \( V_m(Y_t) \) (Shao, Chen and Chen 1998; Rao and Shao 1999; Shao and Chen 1999).

We now consider the estimation of \( V_m(Y_t) \). Under model (2),

\[
    V_m(Y_t) = \sum_k M_k v_{i,k},
\]

which is of the order \( O(N) \), where \( N \) is the size of the population. Usually \( V_m(Y_t) \) is of the order \( O(N^2 / n) \), where \( n = \sum_i n_i \) is the sample size. Hence \( V_m(Y_t) / V_m(Y_t) \) is of the order \( O(n / N) \) and the estimation of \( V_m(Y_t) \) is not necessary if \( n / N \) is negligible (although some sampling fractions \( n_h / N_h \) are not negligible).

In the CES, however, \( n / N \) is around 15% and is not negligible. Hence, the estimation of \( V_m(Y_t) \) is necessary. An asymptotically unbiased and consistent estimator of \( V_m(Y_t) \) is

\[
    v_{i2} = \sum_k M_k s_{k,i}^2,
\]

where \( s_{k,i}^2 \) is the usual sample variance based on the respondents \( y_{i,k} \) in the \( k \)th imputation cell.

### 4. Approximate Variance Estimators

From section 3, a correct variance estimator for \( Y_t \) is \( v_{i1} - v_{i2} \), where \( v_{i1} \) and \( v_{i2} \) are given by (5) and (6), respectively. Although \( v_{i1} \) can be easily extended to the case where \( Y_t \) is replaced by some nonlinear estimator such as \( Y_t^p / Y_t^q \) (the ratio of weekly pay over hour), the extension of \( v_{i2} \) involves the derivation of Taylor expansion for each separate nonlinear estimator. Thus, for the CES, it is desired to derive an approximate variance estimator that is not exactly correct but does not require the computation of \( v_{i2} \).

Note that if \( n / N \) is negligible, then we can simply use \( v_{i1} \) as an estimator of \( V(Y_t - Y_t) \). In the CES, however, using \( v_{i1} \) leads to overestimation, since \( n / N \) is not negligible (see also the simulation results in section 5). Since this overestimation is caused by the sampling fraction, a possible way to fix the problem is to incorporate sampling fractions in the half sample method. When there is no nonresponse, sampling fractions can be incorporated into the half sample method by using formula (2) with \( w_i^{(r)} \) replaced by

\[
    w_i^{(r)} = \begin{cases} 
    (1 + 0.5\sqrt{1 - n_h / N_h}) w_i & \text{if the unit is in the } r \text{th half sample} \\
    (1 - 0.5\sqrt{1 - n_h / N_h}) w_i & \text{if the unit is not in the } r \text{th half sample},
    \end{cases}
\]

when \( i \) is in the stratum \( h \).

Let \( \tilde{v}_{i1} \) be the variance estimator obtained using (5) but with \( w_i^{(r)} \) replaced by \( w_i^{(r)} \). If we use \( \tilde{v}_{i1} \) as an estimator of \( V(Y_t - Y_t) \), however, it has a negative bias, although it is better than the naive estimator that treats imputed values as observed data (see the simulation results in section 5).

While \( v_{i1} \) overestimates and \( \tilde{v}_{i1} \) underestimates the true variance \( V(Y_t - Y_t) \), a compromise is to replace the sampling fraction \( n_h / N_h \) in (7) by the “estimated sampling fraction” \( \hat{r}_{h,t} / N_h \), where \( \hat{r}_{h,t} \) is the number of respondents in stratum \( h \) at month \( t \). Let \( \tilde{v}_{i1} \) be the variance estimator obtained using (5) and (7) but with \( n_h / N_h \) in (7) replaced by \( r_{h,t} / N_h \). Then

\[
    \tilde{v}_{i1} \leq \hat{v}_{i1} \leq v_{i1}.
\]
All three variance estimators are asymptotically unbiased and are approximately equal when $n/N$ is negligible. When $n/N$ is not negligible, however, they are asymptotically biased.

To see the magnitude of the biases $\hat{v}_i$, $\tilde{v}_i$, and $v_i$, we consider the simplest case of no strata and $t = 1$. Let $y_i = y_{i1}, x_i = y_{i0}$ and

$$\hat{Y} = \sum a_i y_i + \sum (1 - a_i) \hat{R} x_i,$$

where $a_i = 1$ if $y_i$ is a respondent and $a_i = 0$ otherwise, $\hat{R} = \sum a_i y_i / \sum a_i x_i$, and all summations are over $i \in S$. Let $\hat{U} = (\sum x_i / n) / (\sum a_i x_i / r)$, where $r$ is the number of respondents. Then the correct variance estimator for $\hat{Y}$ is $v_1 - v_2$ with

$$v_1 = \frac{N^2 \hat{U}^2 s_0^2}{r} + \frac{2N^2 \hat{U} \hat{R} s_{dx} + N^2 \hat{R}^2 s_x^2}{n}$$

and

$$v_2 = N \hat{U} s_0^2 + 2N \hat{U} \hat{R} s_{dx} + N \hat{R}^2 s_x^2,$$

where $s_0^2 = (r - 1)^{-1} \sum a_i (y_i - \hat{R} x_i)^2$, $s_{dx} = (r - 1)^{-1} \sum a_i x_i (y_i - \hat{R} x_i)$, and $s_x^2$ is the sample variance based on $x_i$'s. Also,

$$\tilde{v}_1 = \left(1 - \frac{n}{N}\right) \left(\frac{N^2 \hat{U}^2 s_0^2}{r} + \frac{2N^2 \hat{U} \hat{R} s_{dx} + N^2 \hat{R}^2 s_x^2}{n}\right)$$

$$= v_1 - \frac{nN \hat{U}^2 s_0^2}{r} - \frac{2N \hat{U} \hat{R} s_{dx} + N \hat{R}^2 s_x^2}{n}$$

and

$$\tilde{v}_1 = \left(1 - \frac{r}{N}\right) \left(\frac{N^2 \hat{U}^2 s_0^2}{r} + \frac{2N^2 \hat{U} \hat{R} s_{dx} + N^2 \hat{R}^2 s_x^2}{n}\right)$$

$$= v_1 - N \hat{U} s_0^2 - \frac{2rN \hat{U} \hat{R} s_{dx} + rN \hat{R}^2 s_x^2}{n}.$$ 

Since $v_1 - v_2$ is asymptotically unbiased, the bias of $v_1 = v_i$ is of the same order as $v_2$ and is always non-negative; the bias of $\tilde{v}_1 = \tilde{v}_i$ is of the same order as

$$N \hat{U} s_0^2 \left(1 - \frac{r}{n}\right) N \hat{U} s_0^2 \left(1 - \frac{r}{n}\right),$$

and is always non-positive; and the bias of $\hat{v}_i = \hat{v}$ is of the same order as

$$N \hat{U} (1 - \hat{U}) \left(1 - \frac{r}{n}\right)^2 \left(2N \hat{U} \hat{R} s_{dx} + N \hat{R}^2 s_x^2\right).$$

The bias in (8) is non-negative if $s_{dx} > 0$ and $\hat{U} \approx 1$ (which is true if $a_i$ is independent of $x_i$).

5. Some Simulation Results

To further study the biases of the variance estimators $v_{i1}, \tilde{v}_{i1}$, and $\hat{v}_{i1}$, we conducted a simulation study using a CES dataset (from 1980’s) of 149,044 units as the population $P$. Each unit $i \in P$ has a vector $y_i = (y_{i1}, y_{i2}, y_{i3}, y_{i4}, y_{i5}, y_{i6}, y_{i7})$ and a vector $r_i$ consisting of response indicators of the components of $y_i$, although all values of $y_i$ are available (from administrative records). The sample $S$ in the simulation was obtained by generating a stratified simple random sample $\{y_i\}$ of size 23,092 from $P$ according to the sample allocations listed in Table 1. The response indicators of $\{y_i\}$ in the simulation were generated by drawing another (independent) stratified simple random sample $\{r_i\}$ from $P$. Thus, nonrespondents in the simulation were random and distributed according to the values of the $r_i$’s in the dataset $P$, but independent of the $y_i$’s.

After the sample data and nonrespondents were generated, nonrespondents were imputed as described in section 2. Estimated monthly totals $\hat{Y}_i$ and monthly changes $\hat{Y}_i - \hat{Y}_{i-1}$ were calculated based on imputed data and their variance estimators, $\tilde{v}_{i1}, \tilde{v}_{i2}, v_{i1}$, and $v_{i1} - v_{i2}$ were computed as described in sections 3 and 4. For comparison, the naive variance estimator $v_{i0}$, computed by treating imputed data as observed data, was also computed.

Based on 1,000 simulations, the relative biases (RB) and variances (Var) of the estimated totals $\hat{Y}_i$ and changes $\hat{Y}_i - \hat{Y}_{i-1}$, the RB and coefficient of variations (CV) of the variance estimators for $\hat{Y}_i$ and $\hat{Y}_i - \hat{Y}_{i-1}$, the coverage probability (CP) of the approximate 95% confidence intervals of the form

the estimate $\pm 1.96\sqrt{\text{the estimated variance}},$

and the width (MW) of the confidence interval are given in Tables 2 through 5 respectively for 4 different variables. Estimated simulation standard errors are 2% for RB, CV, and MW, and 0.5% for CP.
### Table 1
Sample Size by Stratum

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<th>Sample Size</th>
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Simulation Results for Employment

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Total: population total.
Change: population difference between the current month and the previous month.

Var: variance of the estimated total or change.
RB: relative bias = 100(bias/true value)%.
CV: coefficient of variation = 100/(standard error/true value)%.
CP: coverage probability of asymptotic confidence interval using estimated variance (in %).
MW: mean width of asymptotic confidence interval $10^7$.
*: Scientific notation (for example, 6,700,000 is 6.7E6).
### Table 3
Simulation Results for Non-supervisory Workers

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Simulation Results for Hours

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Table 5
Simulation Results for Weekly Pay

| Month | Total | RB  | Var | RB  | CV  | CP  | MW  | RB  | CV  | CP  | MW  | RB  | CV  | CP  | MW  | RB  | CV  | CP  | MW  | RB  | CV  | CP  | MW  |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|       |       |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 1     | 2.0E9 | −0.1 | 9.5E12 | 30.7 | 30.4 | 81.8 | 10.3 | 1.7 | 41.9 | 90.0 | 12.4 | 17.2 | 44.3 | 92.4 | 13.3 | 39.8 | 54.4 | 94.4 | 14.6 | 4.3 | 48.9 | 91.0 | 12.6 |
| 2     | 2.1E9 | −0.1 | 1.7E13 | −27.2 | 27.8 | 84.3 | 14.1 | −3.4 | 38.7 | 89.2 | 16.2 | 7.9 | 41.2 | 91.2 | 17.1 | 31.1 | 48.1 | 93.5 | 18.9 | 3.3 | 51.5 | 91.6 | 16.8 |
| 3     | 2.1E9 | −0.1 | 2.2E13 | −14.3 | 34.7 | 85.6 | 17.4 | 1.1 | 42.2 | 88.1 | 18.9 | 8.0 | 43.9 | 89.5 | 19.5 | 34.9 | 51.4 | 93.5 | 21.8 | 2.6 | 50.4 | 91.4 | 19.0 |
| 4     | 2.2E9 | −0.1 | 3.7E13 | −123 | 40.3 | 90.1 | 22.8 | 6.4 | 50.6 | 92.8 | 25.1 | 13.8 | 53.0 | 94.1 | 26.0 | 41.2 | 63.0 | 96.1 | 28.9 | −0.9 | 84.5 | 92.8 | 24.2 |
| 5     | 2.2E9 | −0.1 | 5.0E13 | −16.0 | 41.6 | 89.0 | 25.9 | −1.5 | 51.8 | 91.4 | 28.1 | 5.9 | 54.8 | 92.0 | 29.1 | 29.3 | 64.6 | 94.3 | 32.2 | −5.4 | 56.0 | 92.4 | 27.5 |
| 6     | 2.2E9 | −0.1 | 4.5E13 | −9.4 | 44.1 | 92.0 | 25.5 | −3.8 | 46.9 | 92.6 | 26.3 | 1.8 | 48.7 | 92.8 | 27.1 | 27.8 | 57.8 | 95.0 | 30.3 | 0.4 | 54.1 | 94.2 | 26.8 |
| 7     | 2.2E9 | −0.1 | 3.5E13 | −7.3 | 43.1 | 92.1 | 22.8 | −0.7 | 48.3 | 92.8 | 23.6 | 6.8 | 50.0 | 93.9 | 24.5 | 31.9 | 57.0 | 96.4 | 27.2 | −2.0 | 54.3 | 95.3 | 23.7 |

From Tables 2 through 5, the relative biases of estimators of monthly totals and changes are negligible for all variables. The following is a summary for the simulation results of variance estimators in terms of RB and CV.

1. As expected, the naive variance estimator $v_0$ has a large negative relative bias.

2. The asymptotically unbiased variance estimator $v_1 - v_2$ performs well in general. Its relative bias is always under 10% in absolute value and is frequently under 5%.

3. The variance estimator $v_2$ has a large positive relative bias in all cases. This indicates that the $v_2$ term is not negligible in the CES in which the overall sampling fraction, $n/N$, is about 15%.

4. The variance estimator $\hat{v}_1$, which is the same as $v_1$, but with sampling fractions $n_{by}/N_{by}$ incorporated (section 4), has a negative relative bias in general. Its negative bias may be large, especially in the estimation of the variance for monthly changes.

5. The variance estimator $\hat{v}_{1i}$, which is the same as $\hat{v}_1$ but with sampling fractions $n_{by}/N_{by}$ replaced by $n_{by}/N_{by}$, performs well in a simulation study, although it is not asymptotically unbiased (section 4). Its relative bias is large in a few cases, e.g., in variance estimation for total for weekly pay at months 1 and 4, in variance estimation for total of hours at month 1, and in variance estimation for change of employment at month 7. In many cases, however, the performance of $\hat{v}_{1i}$ is even better than the asymptotically unbiased estimator $v_1 - v_2$.

The following is a summary for the simulation results of confidence intervals in terms of CP and MW.

1. The CP of the confidence interval based on the naive variance estimator $v_0$ is substantially lower than the nominal level 95% in most cases.

2. The CP of the confidence interval based on the asymptotically valid variance estimator, $v_1 - v_2$, is between 90% and 93% in most cases. This is often the case for an asymptotically valid variance estimator, i.e., its relative bias is small but the CP of the related confidence interval is lower than the nominal level. One possible reason is that the convergence in distribution (asymptotic normality, which is the key for asymptotic confidence intervals) requires a larger sample size than the convergence of the second moment (in variance estimation).

3. In terms of CP, the confidence interval based on $v_1$ is the best. This might be because the overestimation in variance offsets the undercoverage in interval estimation. The mean width of the interval based on $v_1$ may
be substantially larger than those of other intervals, especially for weekly pay.

4. The CP of the confidence interval based on $\hat{v}_1$, which is not asymptotically valid, is similar to that of the confidence interval based on $v_{i1} - v_{i2}$.

6. Conclusion and Discussion

For the survey estimators in the Current Employment Survey (CES) with imputed data, we propose an asymptotically unbiased and consistent estimator $v_{i1} - v_{i2}$ (section 3). Although $v_{i1}$ can be easily computed using the grouped balanced half sample method, the computation of $v_{i2}$ involves separate derivations for nonlinear estimators. Thus, several approximations, $v_{i1}$, $\tilde{v}_{i1}$, and $\tilde{v}_{i1}$ (section 4) are considered and compared with $v_{i1} - v_{i2}$ in a simulation study in which a CES dataset is used as population. Our result shows that $v_{i1}$ and $\tilde{v}_{i1}$ have large relative biases, due to the fact that the overall sampling fraction, 15%, is not negligible; the estimator $\tilde{v}_{i1}$, which is the same as $v_{i1}$ but incorporates an estimated sampling fraction (using the rate of response) in the balanced half sample method, performs fairly well. Thus, $\tilde{v}_{i1}$ is recommended to replace $v_{i1} - v_{i2}$ if the computation of $v_{i2}$ is too complicated. Since the use of the “observed sampling fraction” $r_{i1}/N_{it}$ does not reflect the fact that information is available about the non-respondents from previous months, $\tilde{v}_{i1}$ may be improved using a more accurate estimated sampling fraction, for example, Rubin’s (1987) “fraction of missing information”.

Although our study is based on the CES, our results are applicable to any survey that adopts a similar sampling design and a similar imputation method. Furthermore, an extension to the case where model (2) involves $y_{t,i}, y_{t-1,i}, \ldots, y_{t-s,i}$ with an integer $s \geq 2$ is straightforward, although the derivation of $v_{i2}$ (for an asymptotically valid variance estimator) is more complicated.

Acknowledgements

The authors are grateful to an Associate Editor and two referees for their helpful comments and suggestions. The research of Jun Shao was partly supported by the NSF grant DMS-9803112 and DMS-01-02223 and the NSA grant MDA 904-99-1-0032.

Appendix: Proof of (4)

It suffices to show that

$$\text{Cov}_m(\hat{Y}_t, Y_t) = V_m(Y_t).$$

(9)

We show the case of a single imputation cell and $y_{t,i} = y_t^e$ (employment). The general case can be treated similarly.

We use mathematical induction. When $t = 1$,

$$\hat{Y}_t = \hat{\alpha}_1 Y_0,$$

By assumption (2),

$$\text{Cov}_m(\hat{Y}_t, Y_t) = \alpha_1^2 V_m(Y_0) + \sigma^2 E_m(Y_0)$$

$$= N(\alpha_1^2, v_0 + \sigma^2, \mu_0)$$

$$= V_m(Y).$$

Suppose now that (9) is true at time $t - 1$. Let $E_t$, $V_t$ and $\text{Cov}_t$ be the expectation, variance and covariance conditional on $y_{t-1,i}, R_j, j = 1, \ldots, t$. Then

$$E_t(\hat{Y}_t) = \alpha_t \hat{Y}_{t-1},$$

and

$$\text{Cov}_t(\hat{Y}_t, Y_t) = \text{Cov}_t(\hat{\alpha}_t, \hat{Y}_{t-1}, Y_t)$$

$$= \hat{Y}_{t-1} \text{Cov}_t(\hat{\alpha}_t, Y_t)$$

$$= \sigma^2 \hat{Y}_{t-1},$$

where the last equality follows from assumption (2). By the induction assumption,

$$\text{Cov}_m(\hat{Y}_t, Y_{t-1}) = V_m(Y_{t-1}).$$

Then

$$\text{Cov}_m(\hat{Y}_t, Y_t) = \text{Cov}_m[E_t(\hat{Y}_t), E_t(Y_t)] + E_m[\text{Cov}_t(\hat{Y}_t, Y_t)]$$

$$= \alpha_t^2 \text{Cov}_m(\hat{Y}_{t-1}, \hat{Y}_{t-1}) + \sigma^2 E_m(\hat{Y}_{t-1})$$

$$= \sigma_t^2 V_m(Y_{t-1}) + \sigma_t^2 E_m(Y_{t-1})$$

$$= V_m(Y_t).$$

References


