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Price Index Surveys as Quasi-Longitudinal Studies

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ABSTRACT

To calculate price indexes, data on "the same item" (actually a collection of items narrowly defined) must be collected across time periods. The question arises whether such "quasi-longitudinal" data can be modeled in such a way as to shed light on what a price index is. Leading thinkers on price indexes have questioned the feasibility of using statistical modeling at all for characterizing price indexes. This paper suggests a simple state space model of price data, yielding a consumer price index that is given in terms of the parameters of the model.

KEY WORDS: Random walk plus noise model; State space model; Laspeyres index; Paasche index; Geometric price index.

1. INTRODUCTION

Survey sampling for calculation of a consumer price index is characterized by following a given item across time to determine its prices at a succession of times. Only it is not, typically, exactly the same item that is followed – it is not this particular can of Brand Y Tomato Soup at Outlet Z the price of which is repeatedly ascertained, for this particular can is likely to have been sold and consumed, by the time of the next visit of the survey sampler – but rather a succession of items, each fitting the same description ("Brand Y 8 oz. Can of Tomato Soup with Herring, sold at Outlet Z"), the price of which is collected at different times. In other words, it is essentially a group of items fitting a narrow description which is followed across time. For this reason consumer price index surveys may be termed "quasi-longitudinal" as opposed to longitudinal surveys, which would follow individual items across time. Nonetheless, it is reasonable to hope that, having repeated measurements across time might lead to estimation procedures which could capitalize on the time series aspect of such surveys.

In the light of that hope, this paper considers a question which has by and large been ignored by statisticians and economists, or, when not ignored, been answered in the negative: Can a consumer price index (CPI) be treated from a statistical point of view? That is, can the parameter, which characterizes the "change in the cost of living" from one period to another, and which price index surveys attempt to estimate, be defined in terms of a stochastic model?

Aldrich (1992) gives an historic interpretation of early attempts by Jevons and especially Edgeworth, to incorporate distributional assumptions into the CPI. Recent papers on stochastic modeling of the CPI, are those by Balk (1980), Clements and Izan (1981,1987), Bryan and Cechetti (1993), Kott (1984) and Selvanathan and Rao (1994). Diewert (1995) reviews and criticizes these attempts, taking an argument of Keynes (1930) as decisive grounds for rejecting the stochastic approach.

In this paper, a specific approach to modeling the price index using state space models is suggested, and a specific state space model tentatively suggested. This model is applied to scanner data to demonstrate the feasibility of an index based on it. The approach we contemplate, circumvents the Keynesian criticism in fundamental ways, and offers the prospect of the many advantages that sound statistical modeling can bring, including, possibly, simplifications of the survey sampling process.

In what follows, we first briefly review the definition of a price index, and the two (non-stochastic approaches) which have dominated consideration of choice of index (Section 2). We review the Bryan and Cecchetti (1993) example of a statistical model for the price index, and Diewert's formulation of Keynes' objection (Section 3). We then introduce an approach to modeling a consumer price index, that circumvents the Keynes-Diewert difficulties, and that leads naturally to the use of state space models (Section 4). We present results of applying a relatively simple random walk plus noise model to scanner data from the A.C. Nielsen Academic Data Base (section 5). We assess the new index in Section 6, mentioning further research that might be useful.

2. BACKGROUND

What is meant by a Consumer Price Index (CPI) is a single number indicating how the purchasing power of the consumer has changed from one period t' to another t. Its raw ingredients consist of prices for the variety of available items at (at least) the two time periods

$$p_\tau = (p_{\tau 1}, \ldots, p_{\tau N}),\ \tau = t', t$$

as well as quantities of the items sold

$$q_\tau = (q_{\tau 1}, \ldots, q_{\tau N}),\ \tau = t', t.$$

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(Often however in practice quantity data from the periods in question are unavailable, and one makes do with some form of surrogate.) The CPI is derived from a "formula" that uses these raw ingredients:

$$I_{t't} = f(p_{t'}, p_t, q_{t'}, q_t),$$

where $f(\cdot)$ is a function of one of many possible forms. Most such forms have a long history, and have been extensively discussed in the index literature.

As examples, we mention here the Laspeyres index

$$L_{t't} = \frac{\sum_{i=1}^{N} q_{t'i} p_{ti}}{\sum_{i=1}^{N} q_{t'i} p_{t'i}} = \sum_{i=1}^{N} f_{t'i} r_{t't},$$

with $f_{t'i} = q_{t'i} p_{t'i} / \sum_{i=1}^{N} q_{t'i} p_{t'i}$ the "relative expenditures", and $r_{t'i} = p_{ti}/p_{t'i}$ the "price relatives". The Laspeyres index uses the quantities from the earlier time period, as a fixed basis of comparison of the earlier and later prices. The Laspeyres index (or a close variant) has tended to be the index most targeted by governments, because of its simplicity and intelligibility to the layperson.

The natural counterpart to the Laspeyres is the Paasche index

$$P_{t't} = \frac{\sum_{i=1}^{N} q_{ti} p_{ti}}{\sum_{i=1}^{N} q_{ti} p_{t'i}}$$

which standardizes the prices by the later period quantities. Most indices following other formulas will tend to fall between the Paasche and Laspeyres.

For later reference in this paper, we mention an index based on the geometric mean, with fixed non-negative weights f_i, adding to 1:

$$G_{t't} = \prod_{i=1}^{N} \left(\frac{p_{ti}}{p_{t'i}} \right)^{f_i}.$$

This is sometimes referred to as the "Geomean".

Fisher (1922) discusses these and many other index formulae. He introduces what has come to be called the "Test Approach", for deciding among the variety of candidates for the formula $f(\cdot)$: this approach lays out properties ("tests"), which a reasonable index would seem to require, and then examines to what extent each index formula satisfies them.

One of the tests is the Time Reversal Test: $I_{t't} I_{tt'} = 1$. Two indices which continue to exercise their sway in the world, but fail this test are, the Carli-Sauerbach index $C_{t't} = \sum_{i=1}^{N} f_i p_{ti}/p_{t'i}$ and a geomean $\tilde{G}_{t't} = \prod_{i=1}^{N} (p_{ti}/p_{t'i})^{f_{t'i}}$ which employs first period expenditures instead of fixed weights. One readily shows that $C_{t't} C_{tt'} \geq 1$, using the Cauchy-Schwartz inequality, suggesting that this index will run too high.

If an increase in prices on item i tends to give an increase in expenditure share, then $\tilde{G}_{t't} \tilde{G}_{tt'} \leq 1$, so that under such conditions, the first-period-geomean tends to run too low. If an increase in prices on item i tends to give a decrease in expenditure share, then $\tilde{G}_{t't}$ runs too high. In general, we can expect this to be a rather erratic index.

This suggests the following maxim: price indices of the form of a geometric mean, should not have weights tied to prices at one of the periods being compared; those of the form of an arithmetic mean should not have weights independent of those prices.

By contrast with $\tilde{G}_{t't}$, the geomean $G_{t't} = \prod_{i=1}^{N} (p_{ti}/p_{t'i})^{f_i}$ which has fixed weights, is the unique index which satisfies the five axioms on price indices in Balk (1995), and the "circularity test", which says that, for $t' < t^* < t$, $I_{t't} = I_{t't*} I_{t*t}$. Time reversal is an immediate consequence.

Indices which pass most of the tests, tend to be ones incorporating quantity information from both periods; for example, the Fisher index

$$F_{t't} = (L_{t't} P_{t't})^{1/2}$$

and the Törnqvist index

$$T_{t't} = \prod_{i=1}^{N} \left(\frac{p_{ti}}{p_{t'i}} \right)^{f_{t'ti}},$$

with $f_{t'ti} = (f_{t'i} + f_{ti})/2$. The Fisher and Törnqvist are frequently practically indistinguishable. Further discussion of the test approach, may be found in Balk (1995), Diewert (1987), and Eichhorn and Voeller (1976).

The second approach to assessing index formulas is the "economic" approach. This defines a generic index of the form

$$I_{t't} = \frac{C(p_t, U)}{C(p_{t'}, U)},$$

where $U = U(q_1, ..., q_N)$ is a well-defined "utility function", and $C(p_t, U)$ is the minimal cost at prices p_t, of achieving the standard of living, or "utility" U. For a particular utility function U, one inquires whether a particular formula can be regarded as a good approximation to the corresponding cost of living index. Like the test approach, this tends to yield indexes incorporating quantity information from both periods. See Diewert (1987) for further elaboration.

3. THE STOCHASTIC APPROACH

Aldrich (1992) gives the early history of attempts to model price relatives or logarithms of price relatives, using a common parameter that represents the overall rate of growth in prices. A basic theme of his paper is, that the stochastic approach to price indices, while being an early example of the application of statistics to economic concerns, died a natural death. Diewert (1995) also discusses these, as well as more recent examples of the statistical modeling of price relatives. The difficulty which, following Keynes (1930), Diewert finds with such use of models is exemplified by a model of Clements and Izan (1987).

The period from t' to t is divided into equi-temporal pieces, giving relatively short intervals generically represented as being from $t-1$ to t. The logarithm of the price relatives for such a "micro-period", is given by

$$\log\left(\frac{p_{ti}}{p_{t-1,i}}\right) = \pi_t + \beta_i + \varepsilon_{ti}, \qquad (1)$$

with $\varepsilon_{ti} \sim (0, \sigma_t^2/f_i)$. In their model, the f_i's are the average expenditure share of the i-th item over the period t' to t. For the sake of identifiability, it is assumed that $\sum_{i=1}^N f_i\beta_i = 0$. These assumptions lead to a maximum likelihood estimator

$$\hat{\pi}_t = \sum_{i=1}^N f_i \log\left(\frac{p_{ti}}{p_{t'i}}\right),$$

giving an MLE of the price short period price trend as

$$\exp(\hat{\pi}_t) = \prod_{i=1}^N \left(\frac{p_{ti}}{p_{t'i}}\right)^{f_i};$$

that is, based on their stochastic model, one derives a geometric index, with weights f_i, akin to that for the Törnqvist.

Estimates of the β_i and of σ^2 can also be derived, as well as estimates of precision, for example, of the variance of $\hat{\pi}$. Thus, a new statistical foundation seems to be put under an old estimator.

Diewert (1995) raises several objections, none of which can be taken lightly. The chief of these is

> "... the fundamental objection of Keynes (Keynes 1930, p. 78): 'The hypothetical change in the price level [$\exp(\pi_t)$] which should have occurred if there had been no changes in relative prices, is no longer relevant if relative prices have in fact changed – for the change in relative prices has in itself affected the price level'."

If, say, the price of bread relative to the price of automobiles changes, then by that very fact, the overall price level changes.

Keynes' objection is not entirely clear. Why can't there be two aspects of price change, one overall, and the other particular? However, it is not hard to agree that the individual price trends are primary; an overall price trend can only be some weighted sum of these. Diewert offers the following translation into terms of a model, of Keynes' objection. Since we must have the overall price trend of the form

$$\pi_t^* = \sum_{i=1}^N f_i \beta_{ti}^*,$$

the model (1) needs to be replaced by

$$\log\left(\frac{p_{ti}}{p_{t-1,i}}\right) = \pi_t + \beta_{ti} + \varepsilon_{ti}, \qquad (2)$$

with $\beta_{ti} = \pi_t - \beta_{ti}^*$ and $\sum_{i=1}^N f_i\beta_{ti} = 0$. The crucial difference between this and (1) is that now the item parameters β_{ti} are indexed by time. But "then the resulting model has too many parameters to be identified." This would suffice to nullify the approach.

Diewert (1995) does not discuss the much more complicated time-series model of Bryan and Cecchetti (1993). Of preceding papers, it is probably the closest to our present paper, involving a complicated state space model and use of the Kalman Filter. Like the other papers Diewert reviews, it is subject to Keynes' objection.

4. PRICE INDEXES RECONSIDERED

4.1 Common Presuppositions

The stochastic modeling of price behavior given in the last section, whether embodied in equation (1) or (2), or some similar model, has three notable characteristics; the modeling is:

1. *Comprehensive* in the sense that it aims straight for an overall "inflation rate" encompassing all items.
2. *Atomistic*: every item is modelled individually, having its "private" parameter, its own rate of inflation [$\exp(\pi_t + \beta_i)$], apart from all other items.
3. *Time isolated*: price relatives modeling for period $t-1$ to t is disjoint from that for period $t-2$ to $t-1$ *etc.*

It is the combination of these suppositions that yields Diewert's "over-parameterized" argument. The primary thrust of Keynes' criticism is against 1: an overall inflation rate or rise/fall in the cost of living has to be a weighted mixture of several price trends. This may be granted without going so far as to embrace item 2. Item 2 is tacitly accepted in almost all (non-stochastic) constructions of price indices. However, it is not at all clear that every single item has its unique price trend. Different items (for example, Brand X ice cream at several supermarkets) are likely to have a tendency to rise and fall together (at least in the long run). There are degrees of homogeneity between items. In any case, none of these assumptions is a necessary component of a stochastic view of price indices.

4.2 An Elementary State Space Model

We divide the time period t' to t into sub-periods t', $t'+1, ..., t-1, t$, and the collection of heterogeneous items into homogeneous sub-groups g, where the defining characteristic of homogeneity is a tendency to similarity of price change behavior. We make two assumptions:

1. $I_{t't}$ is a mixture of "homogeneous" indices $I_{gt't}$;
2. $I_{gt't}$ can be attained through chaining: $I_{gt't} = \prod_\tau I_{g\tau-1,\tau}$, where $\tau = t'+1, ..., t$.

We focus on a single group index $I_{gt't}$, dropping the subscript g for simplicity of notation. Thus, for the remainder of this paper, we focus on the "sub-index" $I_{t't} \equiv I_{gt't}$.

We proceed to develop an elementary state space model (Harvey 1990, Chapter 3) for the logarithms of the within-group price relatives. Suppose the group contains n items. For $i = 1, ..., n$, let $r_{ti} \equiv p_{ti}/p_{t-1,i}$ be the micro-period price relatives, and $y_{ti} \equiv \log(p_{ti}/p_{t-1,i}) = \log(p_{ti}) - \log(p_{t-1,i})$, their logs. The reason for using logs is that considerable empirical work, beginning with Edgeworth (see Diewert (1995)), suggests that the logs of price relatives will be much closer to having a normal distribution than the price relatives themselves, which can be considerably skewed. Normal distribution of errors is a standard assumption in state space models. Let $y_t \equiv (y_{t1}, ..., y_{tn})$ and $\mathbf{1}$ be a vector of ones of length n.

Consider the multivariate random walk plus noise (RWPN) model

$$y_t = \mathbf{1}\mu_t + \varepsilon_t,\ \varepsilon_t \sim \text{MVN}\left(0, \textstyle\sum_{\varepsilon\varepsilon}\right)$$

$$\mu_t = \mu_{t-1} + \eta_t,\ \eta_t \sim N(0, \sigma_{\eta\eta}) \qquad (3)$$

with $\varepsilon_\tau, \eta_\tau, \tau \in (t', t'+1, ..., t-1, t)$ mutually independent. The model implies that the amount that overall group prices are rising (or falling) in one micro-period, tends to hover around the amount they tended to rise (or fall) in the previous micro-period. This is a matter of common observation: if the price rise in one month tends to be high (low), then in the next month it tends to be correspondingly high (low). Since we are considering a homogeneous set of items, it makes sense that their log price relatives have a common mean. We leave for later work, the question of how to join sub-indices into an overall index.

The model (3) implies the simpler univariate RWPN model

$$\bar{y}_t = \mu_t + \bar{\varepsilon}_t,\ \bar{\varepsilon}_t \sim N(0, \sigma_{\bar{\varepsilon}\bar{\varepsilon}})$$

$$\mu_t = \mu_{t-1} + \eta_t,\ \eta_t \sim N(0, \sigma_{\eta\eta}) \qquad (4)$$

with $\bar{y}_t = n^{-1}1'y_t$, $\bar{\varepsilon}_t = n^{-1}1'\varepsilon_t$, and $\bar{\sigma}_{\varepsilon\varepsilon} = n^{-1}\mathbf{1}'\sum_{\varepsilon\varepsilon}\mathbf{1}$. Some information is thrown away in using (4); on the other hand, the normality assumption is even more likely to hold. For convenience, calculations in the study described in Section 5, were based on the univariate model.

The Kalman Filter (Harvey 1990, Section 3.2) can be used to give estimates $\hat{\mu}_\tau$, and $\hat{\sigma}_{\bar{\varepsilon}\bar{\varepsilon}}, \hat{\sigma}_{\eta\eta}$ of the state parameters μ_τ and the variances $\sigma_{\bar{\varepsilon}\bar{\varepsilon}}, \sigma_{\eta\eta}$ respectively.

Then we define $I_{t't} \equiv E(G_{t't} \mid S_t)$, where $G_{t't} = \prod_i (p_{ti}/p_{t'i})^{f_i}$ is a geomean dependent on fixed shares f_i, and S_t represents the totality of state parameters μ_τ through time t, and also the "hyperparameters" $\sigma_{\bar{\varepsilon}\bar{\varepsilon}}, \sigma_{\eta\eta}$. In other words, we condition on what we take to be the underlying process through time t. Then

$$I_{t't} = \exp\left(\mu_t + \mu_{t-1} + \cdots \mu_{t'+1} + \frac{1}{2}\nu\right), \qquad (5)$$

where $\nu = (t-t')\sum_i\sum_{i'}\sigma_{ii'}f_if_{i'}$, with $\sigma_{ii'}$ the covariance of ε_{ti} and $\varepsilon_{ti'}$ typically of lower order than the state parameters $\mu_{\tau i}$. The natural estimator of $I_{t't}$ is $\hat{I}_{t't} \equiv \exp(\hat{\mu}_t + \hat{\mu}_{t-1} + \ldots \hat{\mu}_{t'+1})$; then

$$E\left(\hat{I}_{t't} \mid S_t\right) = \exp\left(\mu_t + \mu_{t-1} + \cdots \mu_{t'+1} + \frac{1}{2}\tilde{\nu}\right), \qquad (6)$$

where $\tilde{\nu}$, given in the Appendix, does not in general equal ν, but is frequently close, and in any case is of the same order of magnitude. The difference $\Delta(\nu) = \tilde{\nu} - \nu$ can be estimated, by say $\hat{\Delta}(\nu)$, yielding a bias-corrected estimator $\check{I}_{t't} \equiv \hat{I}_{t't}\exp(-1/2\hat{\Delta}(\nu))$. Expressions for ν and $\tilde{\nu}$, and a suggestion for a maximal $\hat{\Delta}(\nu)$, are given in the Appendix. It may be noted that $\hat{\Delta}(\nu)$, and hence $\check{I}_{t't}$, depends on the weights f_i, but that $\hat{I}_{t't}$ does not.

5. EMPIRICAL STUDY

To determine the feasibility of the calculation of price indices using the RWPN model and gain some idea of the behavior of the RWPN index, a small empirical study was made, using price and quantity data for Canned Tuna in the A.C. Nielsen Academic Data Base. Canned tuna has somewhat volatile price and quantity behavior, due to frequent sales, at sometimes very marked discount.

The study covered the Northeast USA and the 104 weeks of the years 1992-1993. The original data set was rather large. To make the investigation manageable, weekly data was combined into 4-week periods, giving a total of 26 periods over two years. Thus for purposes of this study, the data were cumulative quantities and quantity-weighted average prices over four week periods.

The homogeneous groups were defined by brand and type, as follows: 3 brands here labeled A, B, C of "premium" tuna in water, the same three brands of "light" tuna in oil, and again the same three brands of "light" in water, making 9 groups in all.

The study focused on 83 outlets which had positive quantities over most of the 4 week periods, for each of the 9 distinct groups.

The RWPN based index $\hat{I}_{t't}$ and the adjusted RWPN based index $\check{I}_{t't}$ were calculated for four time intervals. In each case, the final period $t = 26$, and the early period was taken successively as $t' = 3, 6, 10, 14$. For the purpose of comparison, we also calculated the corresponding Laspeyres and Paasche Indices. These two standard indices provide also a basis of indirect comparison to the Fisher and Törnqvist, which will be about half way between them.

Figures 1 and 2, for premium and light tuna respectively, give the values of the four indices for the four time intervals, the points representing the state space indices, the lines used to indicate the Laspeyres and Paasche. The adjusted RWPN $\check{I}_{t't}$ is invariably larger than the unadjusted RWPN $\hat{I}_{t't}$. Note that, since it is the first period that we are varying, where the path of indices is monotone up, this would suggest a downward trend in the cost of the particular tuna group (and vice versa).

We observe that the new indices are not out of line with the traditional indices, frequently lying between the Laspeyres and Paasche, but they tend to be considerably more stable as t' changes, suggesting possibly that the traditional indices are reacting to "noise" in the data, and that, in fact, basically very little change is going on in this two year period. It may also be observed in Figure 2, that Light in Oil and Light in Water have similar within brand behavior, suggesting that we might have taken a broader "homogeneous" grouping.

6. FURTHER WORK

The investigation described in this paper suggests several topics for further research.

Measures of precision and estimates of the RWPN indices, in terms of variances or confidence intervals based on the state space model, need to be worked out. Even those who are dubious about the viability of a stochastic methodology in price indices, find the possibility of having a measure of precision appealing (Diewert 1995). It would also be of interest to get measures of precision of more standard indices, based on the state space model.

Empirical work is desirable that investigates more closely what groups of items might best be considered "homogeneous". Also, models possibly more elaborate than the simple RWPN model require investigation. In this respect, the use of scanner data will be a great help, supplying as it does, quantity data as well as prices, in great detail.

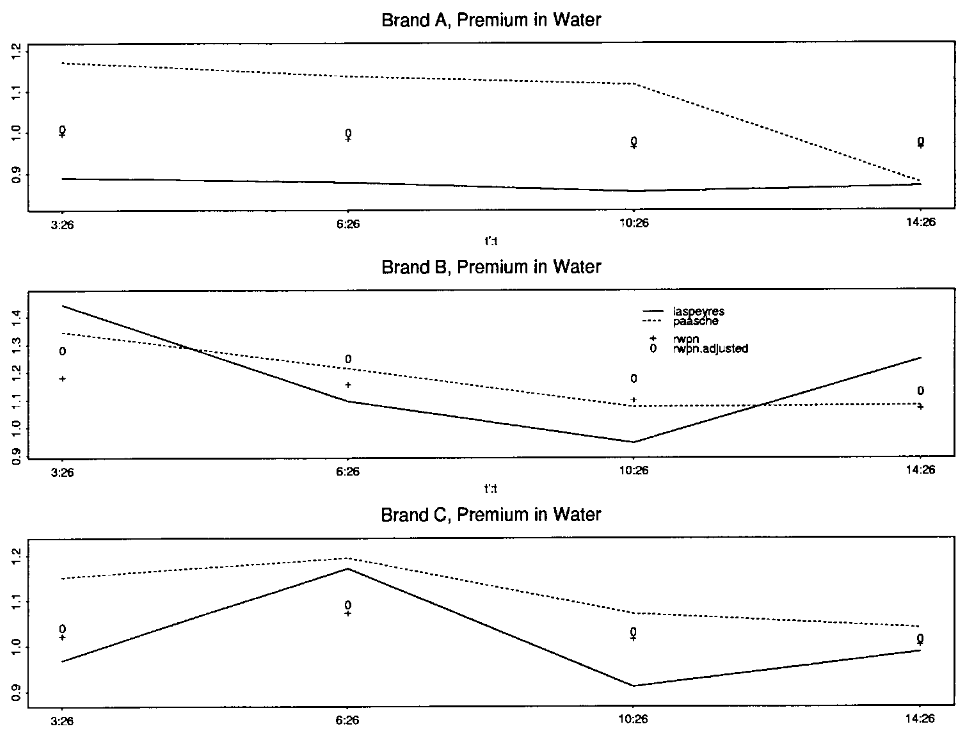

Figure 1. Four Price Indexes for Four Time Intervals, Premium Tuna

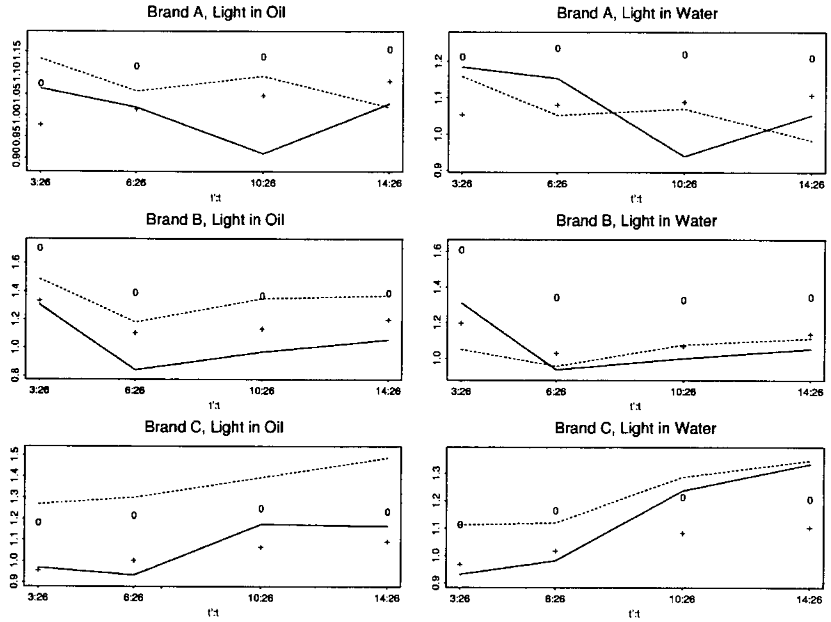

Figure 2. Four Price Indexes for Four Time Intervals, Light Tuna

The state space methodology has methods of handling missing data (Harvey 1990, Section 3.4.7). A point of major concern is how well these models will handle missing data in estimating price indices. In particular, since in practice most data for calculating price indices is based on a small sample of items available, we need to know the robustness of state space indices to the absence of data.

Algorithms for smoothing and forecasting of state space models, are well known. Their use in revising and forecasting indices, might be of great interest.

Finally, in this paper we have focussed only on getting an index for a single homogeneous group. It would be of interest to develop a state space model that combines groups and enables us to get an overall measure of purchasing power.

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APPENDIX

Details of expressions (5) and (6).

We have that

$$G_{t't} = \prod_i \left(\frac{p_{ti}}{p_{t-1,i}} \frac{p_{t-1,i}}{p_{t-2,i}} \cdots \frac{p_{t'+1,i}}{p_{t'i}} \right)^{f_i}$$

$$= \prod_i \left(r_{ti} r_{t-1,i} \cdots r_{t'+1,i}\right)^{f_i},$$

and letting

$$H_{t't} = \log(G_{t't}) = \sum_i f_i \log(p_{ti}/p_{t'i}),$$

we have that

$$H_{t't} = \sum_i f_i \log\left(r_{ti} r_{t-1,i} \cdots r_{t'+1,i}\right)$$

$$= \sum_i f_i \left(y_{ti} + y_{t-1,i} + \cdots y_{t'+1,i}\right)$$

and also that

$$I_{t't} = E(G_{t't}) = \exp(E(H_{t't}) + 1/2 \operatorname{var}(H_{t't})),$$

where the moments are calculated conditional on the state S_t, as in Section 4.3. Let $v = \mathrm{var}(H_{t't})$. Then

$$E(H_{t't}) \equiv E(H_{t't} \mid S_t) = \sum_i f_i(\mu_t + \mu_{t-1} + \cdots \mu_{t'+1}) = \mu_t + \mu_{t-1} + \cdots \mu_{t'+1}$$

and

$$v = \mathrm{var}(H_{t't}) \equiv \mathrm{var}(H_{t't} \mid S_t) = \mathrm{var}\left(\sum_{\tau=t'+1}^{t} \sum_i f_i \varepsilon_{\tau i} \mid S_t \right) = (t-t') \sum_i \sum_{i'} \sigma_{ii'} f_i f_{i'},$$

where $\sigma_{ii'}$ is the covariance of ε_{ti} and $\varepsilon_{ti'}$. We note that $v = (t-t')\sum_i f_i^2 \sigma_{\varepsilon\varepsilon}$, in the special case that the errors ε_{ti} are independent and identically distributed at each time period.

We now consider estimator $\hat{I}_{t't} \equiv \exp(\hat{\mu}_t + \hat{\mu}_{t-1} + \cdots \hat{\mu}_{t'+1})$. We find that $E(\hat{I}_{t't}) = \exp(\mu_t + \mu_{t-1} + \cdots \mu_{t'+1} + 1/2\, \tilde{v})$, where

$$\tilde{v} \equiv \mathrm{var}\left(\sum_{t'+1}^{t} \hat{\mu}_\tau \mid S_t \right) = \left\{ \sum_{t'+1}^{t} \gamma_\tau^2 \right\} \mathrm{var}(\bar{y}_t \mid S_t) +$$

$$\gamma_{t'}^{*2}\, \mathrm{var}(\hat{\mu}_{t'} \mid S_t) = \left\{ \sum_{t'+1}^{t} \gamma_\tau^2 + \gamma_{t'}^{*2} p_{t'-1} \right\} \sigma_{\bar{\varepsilon}\bar{\varepsilon}}$$

with

$$\gamma_\tau = k_\tau \left(1 + \sum_{v=\tau+1}^{t} \prod_{u=\tau+1}^{v} (1-k_u)\right)$$

and

$$\gamma_{t'}^{*} = \sum_{v=t'+1}^{t} \prod_{u=t'+1}^{v} (1-k_u),$$

where

$$k_\tau = p_{\tau|\tau-1} \big/ (p_{\tau|\tau-1} + 1),$$

and $p_{\tau|\tau-1}$, p_τ are the mean square errors of $\hat{\mu}_\tau$ given data up to $\tau - 1$, τ respectively, and are estimated using the Kalman Filter.

This result follows from the equations used in estimating μ_τ:

$$\hat{\mu}_t = k_t \bar{y}_t + (1-k_t)\hat{\mu}_{t-1}$$

$$\hat{\mu}_{t-1} = k_{t-1}\bar{y}_{t-1} + (1-k_{t-1})\hat{\mu}_{t-2}$$

$$\vdots$$

$$\hat{\mu}_{t'+1} = k_{t'+1}\bar{y}_{t'+1} + (1-k_{t'+1})\hat{\mu}_{t'}$$

(*cf.* Harvey 1990, equation 3.2.8), by expressing each $\hat{\mu}_\tau$ in terms of $\bar{y}_\tau, \bar{y}_{\tau-1}, \ldots, \bar{y}_{t'+1}, \hat{\mu}_{t'}$.

In comparing v and $\tilde{v}$, we find, empirically that

$$\sum_{t'+1}^{t} \gamma_\tau^2 + \gamma_{t'}^{*2} p_{t'-1} \approx t - t'.$$

We here consider the simple case where $\mathrm{var}(\varepsilon_{ti}) = \sigma_{\varepsilon\varepsilon}$ and $\mathrm{cov}(\varepsilon_{ti}, \varepsilon_{ti'}) = \rho\sigma_{\varepsilon\varepsilon}$, with $\rho \geq 0$, for $i' \neq i$, that is where not only variances, but all covariances are equal and non-negative. It then can be shown that

$$\sigma_{\bar{\varepsilon}\bar{\varepsilon}} = n^{-2} \sum_i \sum_{i'} \sigma_{ii'} \leq \sum_i \sum_{i'} \sigma_{ii'} f_i f_{i'} \leq n \sum_i f_i^2 \sigma_{\bar{\varepsilon}\bar{\varepsilon}},$$

where n is the number of items in the group. The lower bound is achieved in the case $f_i = 1/n$, and the upper in the case $\rho = 0$. In the first case, no bias adjustment is necessary; in the second, we would take $\hat{\Delta}(v) = \hat{\tilde{v}} - \hat{v}$, where $\hat{v} = (t-t')n\sum_i f_i^2 \hat{\sigma}_{\bar{\varepsilon}\bar{\varepsilon}}$ and $\hat{\tilde{v}} = \{\sum_{t'+1}^{t} \gamma_\tau^2 + \gamma_{t'}^{*2} p_{t'-1}\} \hat{\sigma}_{\bar{\varepsilon}\bar{\varepsilon}}$. These correspond respectively to $\hat{I}_{t't}$ and $\tilde{I}_{t't}$.

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