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A Comparison of Some Estimators of a Set of Population Totals

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ABSTRACT

The Population Estimates Program of Statistics Canada has traditionally been benchmarked to the most recent census, with no allowance for census coverage error. Because of a significant increase in the level of undercoverage in the 1986 Census, however, Statistics Canada is considering the possibility of adjusting the base population of the estimates program for net census undercoverage. This paper develops and compares four estimators of such a base population: the unadjusted census counts, the adjusted census counts, a preliminary test estimator, and a composite estimator. A generalization of previously-proposed risk functions, known as the Weighted Mean Square Error (WMSE), is used as the basis of comparison. The WMSE applies not only to population totals, but to functions of population totals such as population shares and growth rates between censuses. The use of the WMSE to develop and evaluate small-area estimators in the context of census adjustment is also described.

KEY WORDS: Census adjustment; Undercoverage; Small area estimation.

1. INTRODUCTION

The Population Estimates Program of Statistics Canada provides a wide variety of detailed information about the characteristics and distribution of the Canadian population during the five-year period between each census. Intercensal estimates of population have many important uses, including the calculation of billions of dollars of transfer payments from the federal to provincial governments, the estimation of important demographic statistics such as birth and mortality rates, the planning of future levels of immigration, and the weighting of current population surveys such as the monthly Labour Force Survey.

Traditionally, the estimates program is based on the most recent census, with no allowance for coverage error. In the 1986 Census, however, undercoverage increased significantly compared to previous censuses, and continued to be distributed unevenly across geographic and demographic groups. This caused considerable disruption to the estimates program and to the many other programs which use population estimates. As a result, a project was initiated in early 1989 to investigate whether, and if so how, the population estimates in the post-1991 Census period should be adjusted for estimated census coverage error. The research described in this paper was conducted as part of this project. For a more general description of the project, see Royce (1992).

It should be noted that only the population estimates would be affected by this adjustment. The 1991 Census data will be published with no adjustment for undercoverage, other than the small adjustments that have traditionally been made to correct for underenumeration of temporary residents and for persons missed because their dwelling was misclassified as vacant. From the technical point of view, however, the question is quite similar to the issue of census adjustment that has been of interest to many statistical agencies in recent years.

Two key questions in the adjustment issue are the degree to which census counts are improved by adjustment, and which adjustment methods are best. In this paper, we compare the accuracy of several different estimators of a set of population totals, using a weighted mean square error as our criterion.

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Section 2 introduces the topic by considering the simple case of a single population total. We derive and compare the Mean Square Errors of four possible estimators: the unadjusted census count, the adjusted census count, a preliminary test estimator, and a composite estimator. Section 3 extends the results to multiple population totals and to functions of population totals, such as population shares and growth rates. In Section 4, we consider methods for small-area estimation, specifically, the use of synthetic estimation and a special case of synthetic estimation known as across-the-board adjustment. Section 5 concludes with a description of areas for further research.

In developing the estimators described in this paper, two assumptions were made. First, adjustment must result in estimates that are consistent across all geographic and demographic levels, as well as across time. Users consider it to be essential that parts add up to totals, and that there be no major breaks in the time series of estimates. Second, adjustment will be based on the combined results of Statistics Canada's two coverage measurement studies: the Reverse Record Check, which measures gross undercoverage, and the Overcoverage Study, which measures gross overcoverage. Both studies are subject to sampling errors and non-sampling errors.

2. SINGLE POPULATION TOTAL

We first describe and compare four estimators for the case of a single population total. In comparing the estimators, we use the Mean Square Error (MSE) as our criterion.

Let: Y be the known census count;
T be the unknown true population total to be estimated;
U be the true net undercoverage, *i.e.*, $U = T - Y$;
$\hat{U}$ be an estimate of U from the coverage measurement studies;
σ^2 be the variance of $\hat{U}$; and
R be the relative bias of $\hat{U}$, *i.e.* $R = E(\hat{U})/U - 1$.

In the case of all four estimators, our estimate of T can be written as the census count plus some estimate of U. Thus, the MSE of our estimate of T will be the same as that of the corresponding estimate of U. The MSEs (and the WMSEs in later sections) are taken over hypothetical repetitions of the coverage measurement studies, treating the Census counts as fixed quantities.

2.1 Unadjusted Census Estimator

The unadjusted census estimate of U is zero. It has a bias equal to $-U$ and zero variance. Therefore $\text{MSE}(\hat{U}^c) = U^2$.

2.2 Adjusted Census Estimator

The adjusted census estimator of U is $\hat{U}$. It has a bias of UR and a variance equal to σ^2. Thus $\text{MSE}(\hat{U}^A) = \sigma^2 + U^2R^2$.

2.3 Preliminary Test Estimator

A comparison of the MSEs of the previous two estimators suggests that we would use the adjusted census count in preference to the unadjusted census count whenever

$$\sigma^2 < U^2(1 - R^2). \tag{1}$$

Although the parameters in this inequality are unknown, they can (with the exception of R) be estimated from the coverage measurement studies. This suggests the possibility of using these estimates to develop a statistical test of the hypothesis that the inequality holds. The result of the test is then used to choose which estimator to use (thus the term preliminary test, or pretest, estimator).

Specifically, assume that $|R| < 1$, (obviously necessary for (1) to hold) and $\hat{U} \sim N(U(1 + R), \sigma^2)$, where σ^2 is known. Then $\hat{U}^2/\sigma^2$ has a non-central $\chi^2_{(1)}$ distribution with non-centrality paramater $\lambda = U^2(1 + R)^2/2\sigma^2$. The null hypothesis $H_0 : \sigma^2 \geq U^2(1 - R^2)$ is equivalent to the hypothesis $H_0 : \lambda \leq (1 + R)/2(1 - R)$. One approach, therefore, could be to adjust whenever $\hat{U}^2/\sigma^2 > c$, where the critical value $c \geq 0$ is chosen so that

$$\alpha = \Pr\left\{\chi^2\left(1, \frac{1+R}{2(1-R)}\right) \geq c\right\}, \tag{2}$$

where α is the significance level of the test. This is a special case of a more general test suggested by Toro-Vizcorrondo and Wallace (1968).

Note that $\hat{U}^2/\sigma^2$ is the inverse of the square of the estimated coefficient of variation (CV) of $\hat{U}$. Thus, the criterion for adjustment can be interpreted in terms of a requirement to have a sufficiently small (in absolute value) CV.

In practice, we would have to substitute some prior estimate of the relative bias, say r, for R in (2). The sensitivity of c to various values of R is examined in Royce (1991) for the case of a one-sided test (a normal distribution was used instead of a χ^2 in this case). For example, with a significance level of 2.5%, it was found that the critical CV was only reduced from 33.8% to 27.1% even when the relative bias was as much as 50%.

If σ is not known but an estimate $\hat{\sigma}$ is available, then a similar test can still be constructed by assuming that

$$\frac{\nu\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{(\nu)} \tag{3}$$

independent of $\hat{U}$. This leads to a test based on a non-central F distribution. Further details on the construction of such tests are given in Judge and Bock (1978).

In order to determine the MSE of the preliminary test estimator, we note that it can be written as $\hat{U}^P = I\hat{U}$ where

$$\begin{aligned} I &= 1 \quad \text{if} \quad \frac{\hat{U}^2}{\sigma^2} > c \\ &= 0 \quad \text{if} \quad \frac{\hat{U}^2}{\sigma^2} \leq c. \end{aligned} \tag{4}$$

When σ^2 is known, the MSE of this estimator can be shown to be (see, for example, Judge and Bock 1978, p. 72)

$$\begin{aligned} \text{MSE}(\hat{U}^P) = \sigma^2 + U^2R^2 + (2U^2(1 + R) - \sigma^2)\Pr\{\chi^2_{(3,\lambda)} \leq c\} - \\ U^2(1 + R)^2\Pr\{\chi^2_{(5,\lambda)} \leq c\}. \end{aligned} \tag{5}$$

Note that as $c \to \infty$, *i.e.* as the chance of adjustment goes to zero, the MSE approaches U^2, the MSE of the unadjusted census. Similarly, as $c \to 0$, *i.e.* as the chance of adjustment goes to certainty, the MSE approaches $\sigma^2 + U^2R^2$, the MSE of the adjusted census estimator. Thus, the two previous approaches of adjustment or no adjustment can be interpreted as extreme cases of the pretest estimator procedure.

Figure 1 shows MSE/σ^2 for the preliminary test estimator as a function of U^2/σ^2, for various values of c, in the unbiased case ($R = r = 0$). The MSEs/σ^2 of the unadjusted census and the adjusted census are also shown. In all cases, the MSE of the preliminary test estimator starts out higher than that of the unadjusted census, crosses the MSE of the adjusted census, reaches a maximum, and then approaches the MSE of the adjusted census. As the value of c decreases and the level of significance α of the test therefore increases, the MSE of the preliminary test estimator approaches that of the adjusted census more quickly, but at the expense of being higher for small values of U^2/σ^2. Thus, the performance of the preliminary test estimator over the range of possible values of U^2/σ^2 depends on the level of significance that is chosen for the test.

Figures 2 and 3 show similar plots in the case where $R = .5$ and $R = -.5$ respectively (since we may feel we have no information on which to base an estimate of R, we have set $r = 0$). Again, the MSEs of the preliminary test estimators approach those of the adjusted census as U^2/σ^2 increases. With a positive bias the MSE of the preliminary test estimator approaches the MSE of the adjusted census more quickly than in the unbiased case, while for a negative bias the reverse is true.

What is the "best" value of c for the test? Ideally, we would like to choose c so that the MSE of the preliminary test estimator is as close as possible to the minimum of the MSEs of the adjusted census and the unadjusted census. One approach, due to Sawa and Hiromatsu (1973) and extended by Brook (1976), is to minimize the maximum difference between the MSE of the preliminary test estimator and the minimum of the MSEs of the adjusted census and unadjusted census. For the unbiased case this criterion gives an optimal value of c of approximately 1.88. This corresponds to a critical CV (in absolute value) for the estimated undercoverage of 73%. The MSE of this estimator is shown in Figure 4.

Judge and Bock (1978) also describe other approaches to choosing the optimal value of c, such as minimizing the average distance (rather than the maximum difference) and Bayesian approaches.

2.4 Composite Estimator

The preliminary test estimator was written as $\hat{U}^P = I\hat{U}$, where I took on only the values 0 or 1. Because of this inherent discontinuity, however, it has been shown that the preliminary test estimator is inadmissible (Cohen 1965). As an alternative, we might consider letting the multiplier of $\hat{U}$ take any value between 0 and 1. That is, instead of using the data to tell us **whether** to adjust, we use the data to tell us **how much** to adjust. This type of estimator has been suggested by Spencer (1980) and more recently by Andrews (1991). We define $\hat{U}^\alpha = \alpha\hat{U}$ where $0 \leq \alpha \leq 1$. For a given alpha, this estimator has a MSE equal to

$$\mathrm{MSE}(\alpha\hat{U}) = \alpha^2\sigma^2 + U^2(\alpha(1 + R) - 1)^2, \tag{6}$$

which is minimized when

$$\alpha = \frac{U^2(1 + R)^2}{(1 + R)(\sigma^2 + U^2(1 + R)^2)}. \tag{7}$$

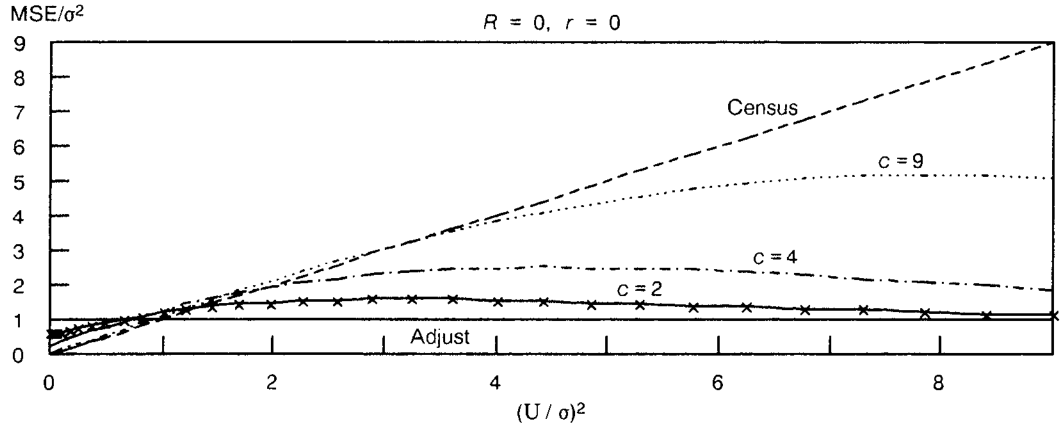

Figure 1 Comparison of MSEs

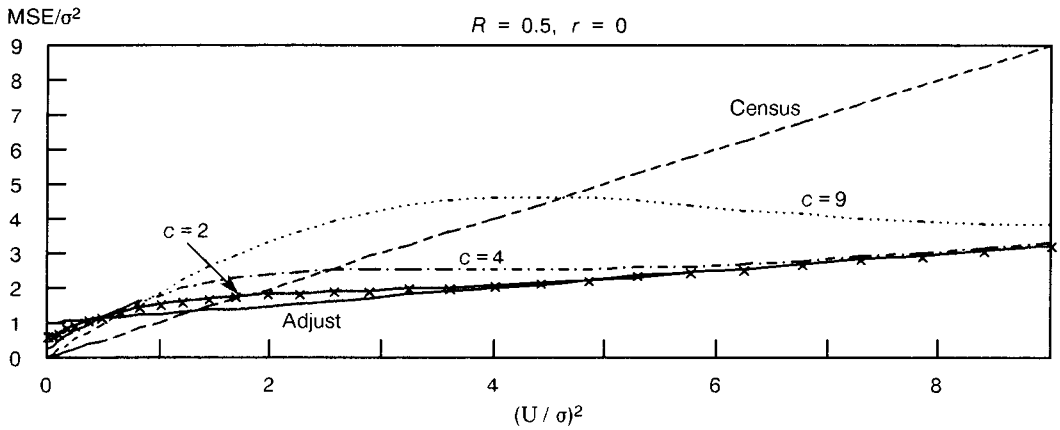

Figure 2 Comparison of MSEs

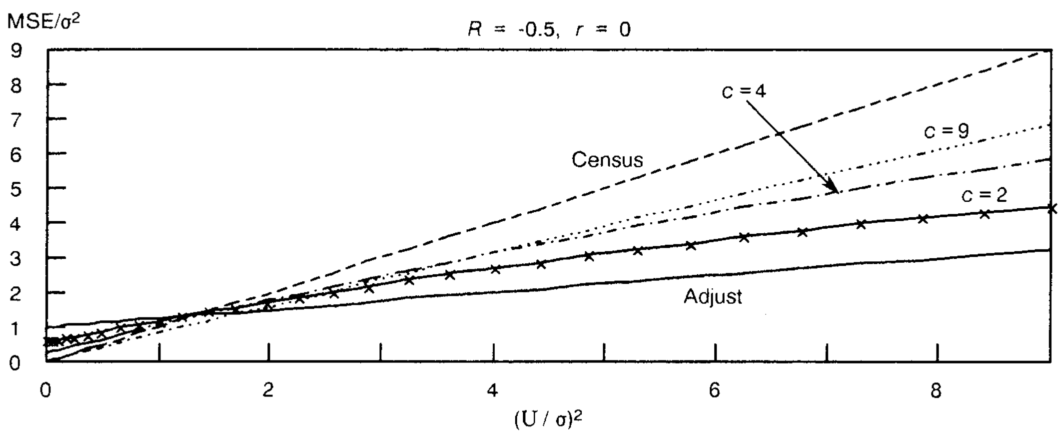

Figure 3 Comparison of MSEs

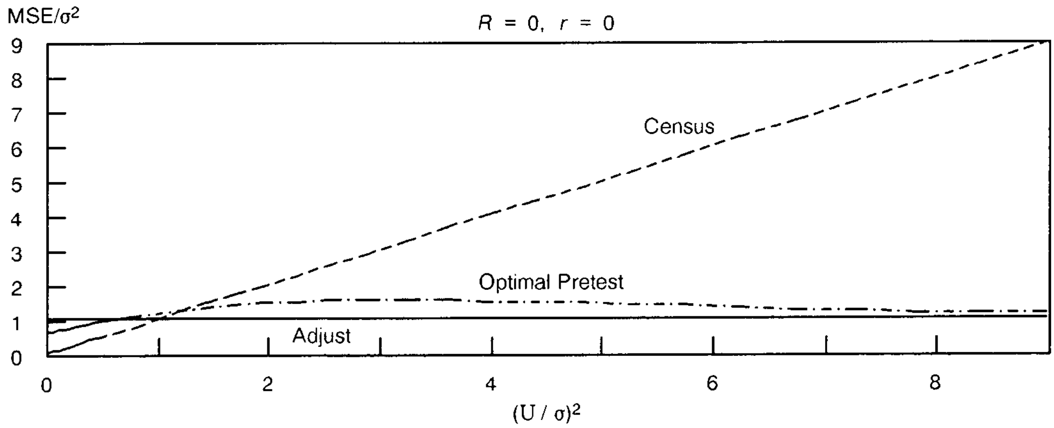

Figure 4 Comparison of MSEs

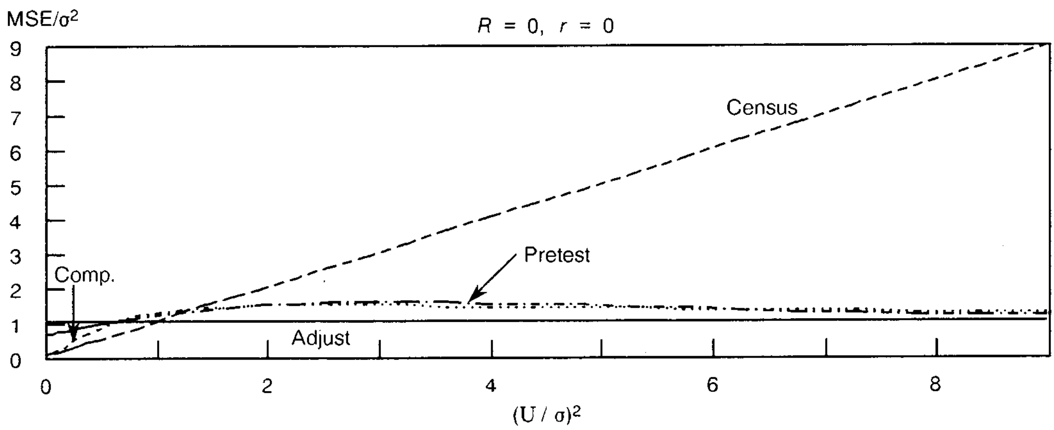

Figure 5 Comparison of MSEs

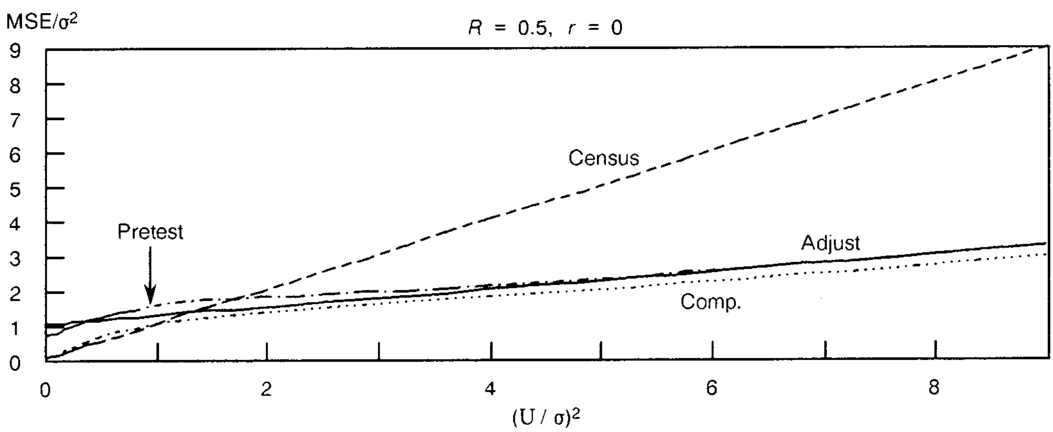

Figure 6 Comparison of MSEs

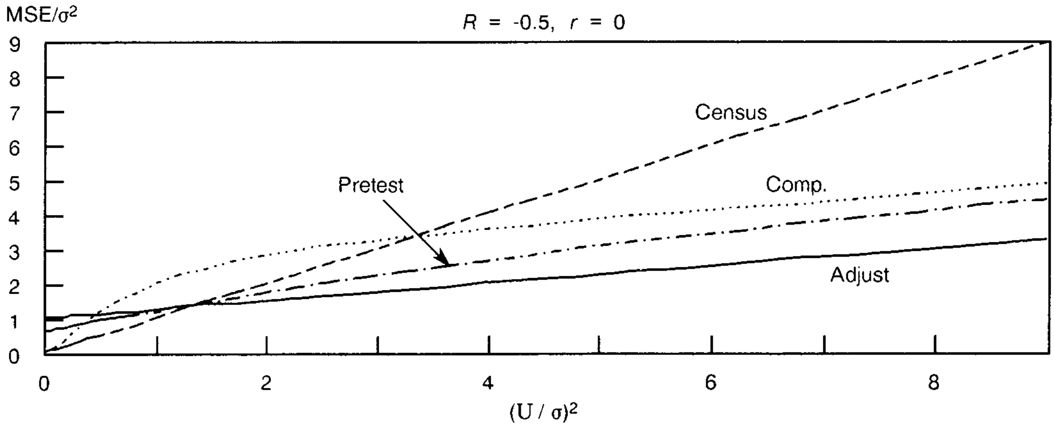

Figure 7 Comparison of MSEs

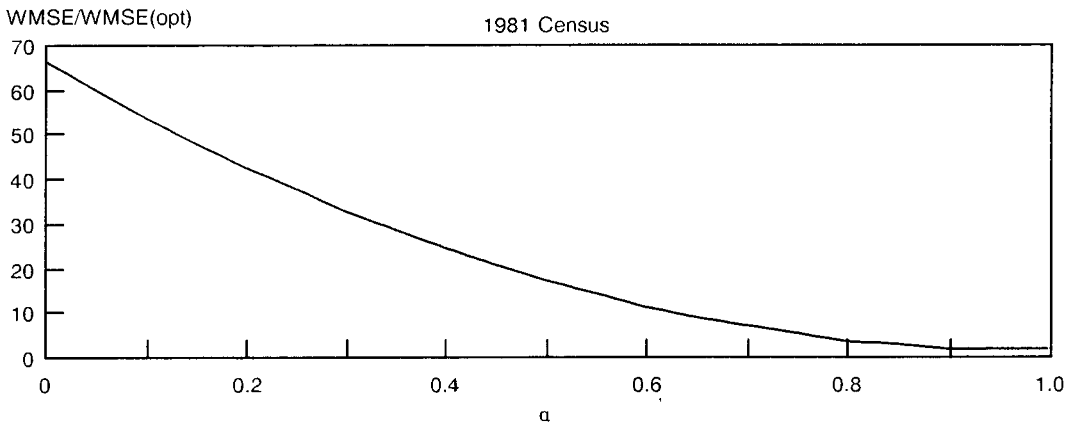

Figure 8 WMSEs for Totals

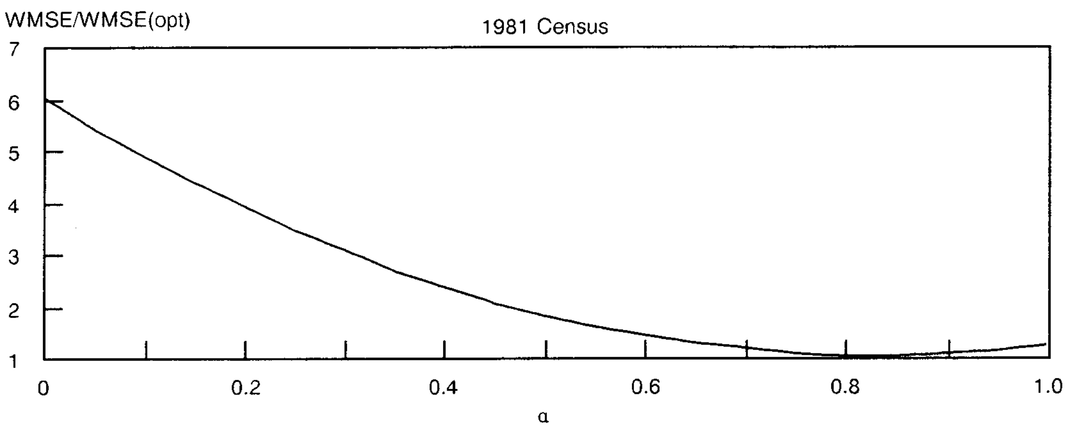

Figure 9 WMSEs for Shares

If σ is assumed known, then a possible estimator of α is

$$\hat{\alpha} = \frac{\hat{U}^2}{(1 + r)(\sigma^2 + \hat{U}^2)} \tag{8}$$

and thus

$$\hat{U}^{\hat{\alpha}} = \frac{\hat{U}^3}{(1 + r)(\sigma^2 + \hat{U}^2)}. \tag{9}$$

The approximate MSE of this estimator can be found using a Taylor series approximation. Letting

$$h(U,\sigma^2) = \frac{U^3}{(1 + r)(\sigma^2 + U^2)} \tag{10}$$

we get (dropping terms higher than those involving the first derivative)

$$\text{MSE}(\hat{U}^{\hat{\alpha}}) \doteq (h(U,\sigma^2) - U)^2 + \left(\frac{\partial h(U,\sigma^2)}{\partial U}\right)^2 (U^2R^2 + \sigma^2)$$

$$+ 2(h(U,\sigma^2) - U)\left(\frac{\partial h(U,\sigma^2)}{\partial U}\right) UR. \tag{11}$$

This approximation can also be extended to the case where σ is unknown by making the assumption given in (3). The MSE is then increased by the additional term

$$\left(\frac{\partial h(U,\sigma^2)}{\partial \sigma^2}\right)^2 \frac{2\sigma^4}{\nu}. \tag{12}$$

Figures 5, 6 and 7 show the MSE of the composite estimator as a function of U^2/σ^2, as well as the MSEs of the unadjusted census, adjusted census, and the optimal preliminary test estimator from Section 2.3. In the unbiased case (Figure 5) and the positive bias case (Figure 6), the composite estimator outperforms the optimal preliminary test estimator. When the bias is negative, however, (Figure 7) the MSE of the composite estimator can be much higher than any of the other estimators over a considerable portion of the range of U^2/σ^2.

3. MORE GENERAL ESTIMATORS

In this section, we generalize the four estimators examined in Section 2 in two ways. First, instead of a single population total, we consider a vector of population totals, denoted as $\underline{T} = (T_1, T_2, \ldots, T_N)$. Second, we consider not only the population totals themselves, but also functions of the population totals, denoted by $\underline{g}(\underline{T}) = (g_1(\underline{T}), g_2(\underline{T}), \ldots, g_K(\underline{T}))$ where in general $K \neq N$. Typical functions of interest include population shares, used in the transfer of funds from the federal to provincial governments, as well as growth rates between censuses, differences in growth rates among different provinces, and so on.

In evaluating the overall accuracy of some estimate $\underline{g}(\hat{\underline{T}}^*)$ for $\underline{g}(\underline{T})$, we will make use of a loss function. The use of loss functions for evaluating the effects of census adjustment is described in Fellegi (1980), Citro and Cohen (1985), Spencer (1986), and Wolter and Causey (1991) to name just a few. The specific loss function used in this paper is a generalization of previously-proposed loss functions for population totals and shares. Specifically, the risk (expected loss) of the estimator $\underline{g}(\underline{\hat{T}}^*)$ is the Weighted Mean Square Error, defined as

$$\text{WMSE}\,(\underline{g}(\underline{\hat{T}}^*)) = E\left\{\sum_{k=1}^{K} w_k\big(g_k(\underline{\hat{T}}^*) - g_k(\underline{T})\big)^2\right\}, \tag{13}$$

where w_k is a user-specified weight reflecting the importance of the k-th component of the loss function.

Since $\underline{g}$ may be complex in practice, it is useful to work instead with an approximation to the WMSE derived by expanding $\underline{g}(\underline{\hat{T}}^*)$ in a Taylor series around $\underline{T}$. This yields:

$$\text{WMSE}\underline{g}(\underline{\hat{T}}^*) \doteq \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_{ij}[\text{Cov}(\hat{U}_i^*, \hat{U}_j^*) + \text{Bias}(\hat{U}_i^*)\text{Bias}(\hat{U}_j^*)] \tag{14}$$

where the weight ω_{ij} is given by

$$\omega_{ij} = \sum_{k=1}^{K} w_k \frac{\partial g_k}{\partial T_i}\frac{\partial g_k}{\partial T_j}. \tag{15}$$

(Note that the approximate WMSE can also be written as the expectation of the quadratic form $(\underline{\hat{U}}^* - \underline{U})'\Omega(\underline{\hat{U}}^* - \underline{U})$ where ω_{ij} is the ij-th element of Ω.)

This formulation conveniently splits each component of the risk function into two parts: a weight ω_{ij} that depends only on the w_k and the function $\underline{g}$, and the portion in square brackets which depends only on the particular estimator being used.

While the choice of the w_k can be arbitrary, considerations of equity have often led to the choice $w_k = 1/T_k$. In the case of population totals and shares, for example, the risk function (14) then becomes equivalent to those proposed by Fellegi (1980) and also used by Wolter and Causey (1991), among others. Other choices for the weights that have been suggested in the literature include $w_k = 1/Y_k$, $w_k = 1/\hat{T}_k$, and $w_k = 1$. For further discussion on the merits of these various weightings, see the references cited above. Table 1 shows some examples of ω_{ij} for different functions.

In the case of population growth rates, the first pair of subscripts on the omega refer to the population quantity of interest (*e.g.* province) while the second pair refer to the census at time 1 or time 2 respectively. The second subscript on the T_i also refer to the census at time 1 or 2.

In the remainder of this section, we illustrate the use of the WMSE in developing and evaluating the unadjusted census, adjusted census, preliminary test estimator, and composite estimator.

3.1 Unadjusted Census

The WMSE of the unadjusted census is $\text{WMSE}(\underline{\hat{U}}^C) = \sum_{ij}\omega_{ij}U_iU_j$.

3.2 Adjusted Census

The WMSE of the adjusted census is $\text{WMSE}(\underline{\hat{U}}^A) = \sum_{ij}\omega_{ij}[\sigma_{ij} + b_ib_j]$ where $\sigma_{ij} = \text{Cov}(\hat{U}_i,\hat{U}_j)$ and $b_i = \text{Bias}(\hat{U}_i)$.

Table 1

Examples of Weights ω_{ij} in the Approximate WMSE for Various Functions

Function	ω_{ij}
Set of Population Totals	$\omega_{ii} = w_i$ $\omega_{ij} = 0 \quad i \neq j$
Set of Population Shares	$\omega_{ii} = \frac{1}{T^4}\left(\sum_k w_k T_k^2 + w_i T^2 - 2w_i T T_i\right)$ $\omega_{ij} = \frac{1}{T^4}\left(\sum_k w_k T_k^2 - T(w_i T_i + w_j T_j)\right) \quad i \neq j$
Set of Growth Rates	$\omega_{ii11} = \frac{w_i T_{i2}^2}{T_{i1}^4}$ $\omega_{ii12} = -\frac{w_i T_{i1} T_{i2}}{T_{i1}^4} = \omega_{ii21}$ $\omega_{ii22} = \frac{w_i T_{i1}^2}{T_{i1}^4}$ $\omega_{ij11} = \omega_{ij12} = \omega_{ij21} = \omega_{ij22} = 0 \quad i \neq j$

3.3 Preliminary Test Estimator

As in Section 2.3, we would use the adjusted census in preference to the unadjusted census if the WMSE of the adjusted census is less than the WMSE of the unadjusted census, *i.e.*, if

$$D = \sum_{ij} \omega_{ij}[U_i U_j - \sigma_{ij} - b_i b_j] > 0. \tag{16}$$

Tests for this type of hypothesis were suggested by Fellegi (1980) for the specific cases of population totals and population shares, but the ideas generalize quite readily to any function $\underline{g}$. The left hand side of the inequality (16) is estimated by $\hat{D} = \sum_{ij} \omega_{ij}[\hat{U}_i \hat{U}_j - 2\sigma_{ij}]$ where the ω_{ij} are assumed to be known. (In practice the ω_{ij} are estimated by substituting either the census counts or the adjusted census counts in (13). Fellegi claimed that minor variations in the weights were unlikely to substantially change the test results.) It is then easy to show that $E(\hat{D}) = D + 2\sum_{ij} \omega_{ij} b_i (U_j + b_j)$. For the case of totals and shares, Fellegi presented arguments why it could be assumed that the second term was non-positive, *i.e.*, $\sum_{ij} \omega_{ij} b_i (U_j + b_j) \leq 0$ so that $\hat{D}$ would tend to underestimate D. Fellegi also derived an approximate variance for $\hat{D}$. This, along with the assumption that $\hat{D}$ was normally distributed, permitted the construction of a test for the hypothesis given in (16).

Table 2

z Values for Fellegi's Tests for Adjustment of Provincial Population Totals and Shares, Reverse Record Check, 1976, 1981 and 1986

Function	1976	1981	1986
Totals	9.3	10.1	13.1
Shares	3.1	1.8	1.5

In the more general case, the approximate variance of $\hat{D}$ is given by $\mathrm{Var}(\hat{D}) \doteq 4\sum_{ij}\sigma_{ij}(\sum_{i'j'}\omega_{ij'}\omega_{i'j}U_{i'}U_{j'})$. An estimate of $\mathrm{Var}(\hat{D})$ can then be derived by substituting estimates of the U_i and σ_{ij} in this formula.

In the case of totals, for example, the test statistic (z value) is given by

$$z = \frac{\hat{D}}{\sqrt{\hat{\mathrm{Var}}(\hat{D})}} = \frac{\sum_i \frac{\hat{U}_i^2 - 2\sigma_i^2}{Y_i}}{2\sqrt{\sum_i \frac{\hat{U}_i^2\hat{\sigma}_i^2}{Y_i^2}}}, \tag{17}$$

where in this case the inverse of the census counts have been used as the weights. A similar expression can be derived for population shares.

Table 2 shows the z values calculated for the censuses of 1976, 1981 and 1986 for provincial population totals and shares. The data come from the Reverse Record Checks conducted in these censuses.

The case for adjusting population totals is much stronger than the case for adjusting shares, reflecting the fact that estimates of differences in undercoverage rates among provinces are less accurate than estimates of the undercoverage rates themselves. Further numerical results are given in Royce and Luc (1990).

3.4 Composite Estimator

A natural extension of the composite estimator of Section 2.4 would at first seem to be $\alpha_i\hat{U}_i$. However the use of different amounts of adjustment for each value of i introduces problems of consistency. For example, it would imply that more adjustment should be done at the Canada level than at the province level, since the estimates of undercoverage at the province level will be less accurate than for the national level. If this were done, the provincial totals would not add up to the Canada total.

In practice, therefore, we constrain ourselves to a single value of alpha, *i.e.* $\underline{\hat{U}}^\alpha = \alpha\underline{\hat{U}}$, where again $0 \leq \alpha \leq 1$. The WMSE of this estimator is

$$\mathrm{WMSE}(\underline{\hat{U}^\alpha}) = \sum_{ij}\omega_{ij}[\alpha^2(\sigma_{ij} + b_ib_j) + (\alpha - 1)^2U_iU_j + 2\alpha(\alpha - 1)U_ib_j], \tag{18}$$

which is minimized when

$$\alpha = \frac{\sum_{ij} \omega_{ij} U_i (U_j + b_j)}{\sum_{ij} \omega_{ij} [\sigma_{ij} + (U_i + b_i)(U_j + b_j)]}. \tag{19}$$

If, as was done in Section 3.3, we make the assumption that $\sum_{ij} \omega_{ij} b_i (U_j + b_j) \leq 0$ then a lower bound for the optimal alpha is given by

$$\alpha_L = \frac{\sum_{ij} \omega_{ij} (U_i + b_i)(U_j + b_j)}{\sum_{ij} \omega_{ij} [\sigma_{ij} + (U_i + b_i)(U_j + b_j)]}, \tag{20}$$

which we estimate by

$$\hat{\alpha}_L = \frac{\sum_{ij} \omega_{ij} \hat{U}_i \hat{U}_j}{\sum_{ij} \omega_{ij} [\hat{\sigma}_{ij} + \hat{U}_i \hat{U}_j]}, \tag{21}$$

assuming the ω_{ij} are known. In practice, as we did for the preliminary test estimator, we would estimate the ω_{ij} by substituting census counts or adjusted census counts in (15).

In the case of population totals, for example, the estimated amount of adjustment is

$$\hat{\alpha}_L = \frac{\sum_i \hat{T}_i \hat{U}_i^2}{\sum_i \hat{T}_i [\hat{\sigma}_i^2 + \hat{U}_i^2]}, \tag{22}$$

where $\hat{U}_i$ is the estimated undercoverage **rate**, *i.e.* $\hat{U}_i / (Y_i + \hat{U}_i)$, and $\hat{\sigma}_i^2$ is its estimated variance.

For shares, the amount of adjustment is given by

$$\hat{\alpha}_L = \frac{\sum_i \hat{T}_i \hat{U}_i^2 - \hat{T} \hat{U}^2}{\sum_i \hat{T}_i [\hat{\sigma}_i^2 + \hat{U}_i^2] - \hat{T}(\hat{\sigma}^2 + \hat{U}^2)}, \tag{23}$$

where $\hat{U}$ is the estimated undercoverage rate for the total population, *i.e.* $\sum_i \hat{U}_i / \sum_i (Y_i + \hat{U}_i)$ and $\hat{\sigma}^2$ is its estimated variance. The inverse of the adjusted census counts have been used as the weights in these two examples.

3.5 Numerical Comparisons

In the case of a single population total, it was possible to derive exact or approximate formulae for the MSEs of the four estimators as a function of U^2/σ^2, R, r and (in the case of the preliminary test estimator), the critical value of the test. Unfortunately, it has not yet been possible to derive similar expressions for the WMSEs of complex functions of a vector of population totals.

In the case of the unadjusted census, adjusted census, and composite estimator, however, it is possible to estimate the WMSEs by substituting estimates of undercoverage and their estimated variances into equation (18) (if estimates of the bias terms are available they can be used, but in what follows we assume they are zero). For example, Figures 8 and 9 show, for the 1981 Census, the estimated ratio of the WMSE to the optimal WMSE, as a function of α, where the provinces are again the units indexed by i. The extremes of $\alpha = 0$ and $\alpha = 1$ correspond to the unadjusted and adjusted census counts respectively, while the minimum point on the curve corresponds to the optimal α. Figure 8 is for totals and Figure 9 is for shares. The optimum values of α were computed using formulae (22) and (23).

In each case, the optimal degree of adjustment is close to 1.0, and results in a WMSE considerably lower than the WMSE corresponding to no adjustment (*e.g.* by a factor of almost 70 for totals). The optimal degree of adjustment is less for shares than for totals, again reflecting the fact that estimates of differences in coverage rates between provinces are less accurate than the estimates of the rates themselves. It is also interesting to note that the WMSE for full adjustment is only slightly higher than that of the optimal degree of adjustment. This can have important practical significance, since it is much easier to explain a full adjustment to data users than to explain a partial adjustment.

4. SMALL AREA ESTIMATION

The previous two sections considered the case where direct estimates of undercoverage, and estimates of their variances, were available from the coverage measurement studies. This situation applies, for example, for provinces, for some major Census Metropolitan Areas, and for broad demographic groups (*e.g.* age by sex, age by marital status) at the national level. However the Population Estimates Program produces estimates at very detailed levels, such as single years of age by sex by marital status for some 260 Census Divisions. Direct estimates of undercoverage generally do not exist at such levels.

Nevertheless, the need to maintain consistency of the estimates requires that any adjustment made at a higher level be "carried down" to the detailed levels used by the estimates program. In this section, we consider the use of synthetic estimation for this purpose, and show how the WMSE can again be used to develop preliminary test estimators and composite estimators.

The synthetic estimator is based on the assumption that net undercoverage is uniform within each of a number of "adjustment groups", indexed by a. The synthetic estimate is then given by $\hat{U}_i^S = \sum_a \lambda_{ia} \hat{U}_a$ where $\lambda_{ia} = Y_{ia}/Y_a$. For example, the adjustment groups might correspond to age-sex groups, for which estimates of undercoverage $\hat{U}_a$ are available at some higher level.

A special case of the synthetic estimator arises when there is only one adjustment group. Wolter and Causey (1991) have called this the across-the-board estimator. It is defined as $\hat{U}_i^{ATB} = \lambda_i \hat{U}$ where $\lambda_i = Y_i/Y$. WMSEs for the across-the-board and the synthetic estimator can be derived using equation (14). Since the ω_{ij} do not depend on the particular estimator used, only the portion in square brackets changes. Table 3 compares the estimators of U_i and their covariance and bias terms for the census, adjusted census, across-the-board and synthetic estimators.

Table 3

Examples of Covariance and Bias Terms in the Approximate WMSRE for Various Estimators

Estimator	$\hat{U}_i^*$	$\mathrm{Cov}(\hat{U}_i^*,\hat{U}_j^*)$	$\mathrm{Bias}(\hat{U}_i^*)$
Census	0	0	$-U_i$
Adjusted Census	$\hat{U}_i$	σ_{ij}	b_i
Across-the-Board	$\lambda_i\hat{U}$	$\lambda_i\lambda_j\sigma^2$	$\lambda_i(U+b)-U_i$
Synthetic	$\sum_a \lambda_{ia}\hat{U}_a$	$\sum_{aa'} \lambda_{ia'}\lambda_{ja'}\sigma_{aa'}$	$\sum_a \lambda_{ia}(U_a+b_a)-U_i$

where $b = \sum_i b_i$ and similarly b_a is the bias of $\hat{U}_a$.

4.1 Preliminary Test Estimators

As was the case in Sections 2 and 3, the WMSE can be used to develop statistical tests to decide between two competing estimators. As an example, consider the situation where we wish to choose between the unadjusted census and the across-the-board estimator for population totals (shares are of course unchanged by across-the-board adjustment). On comparing the WMSEs of these two estimators, we find that we would use the across-the-board estimator in preference to the census counts if

$$\sigma^2 < U^2(1-R^2)\left[1-\frac{2TB}{U(1-R)}\right], \tag{24}$$

where

$$B = 1 - \frac{1}{\sum_i \frac{\lambda_i^2}{\tau_i}} \tag{25}$$

and $\tau_i = T_i/T$. This condition was given, in a different form, by Wolter and Causey (1991). B is a measure of the heterogeneity of undercoverage; it is non-negative, and is equal to zero if and only if the undercoverage is completely uniform.

Noting that this inequality is the same as (1) except for the additional term in square brackets, we can derive a test very similar to the test described in Section 2.3. The critical value of the coefficient of variation will depend on the chosen significance level and the relative bias as before, but will also depend on $B/\bar{U}$, the ratio of the heterogeneity of undercoverage to the overall undercoverage rate.

Royce (1991) showed that, in practice, the effect of this additional factor on the critical CV was likely to be negligible. Thus, if adjustment is justified at some higher level, then carrying down the adjustment to lower levels is almost certainly justified as well. Similar results were found in a simulation study reported by Wolter and Causey (1991).

4.2 Composite Estimators

In Sections 2 and 3 we considered composite estimators where the two extremes were the unadjusted census and the adjusted census. With the addition of the synthetic and across-the-board estimators, the number of possible composite estimators increases considerably. For example, we might consider composite estimators involving the unadjusted census and the synthetic estimator, the adjusted census and the across-the-board estimator, the across-the-board estimator and the synthetic estimator, and so on. Consequently, we present below a method which can be used to derive a composite estimator involving any two estimators.

Our general composite estimator is defined as $\underline{\hat{U}^*} = \alpha\underline{\hat{U}_1} + (1-\alpha)\underline{\hat{U}_2}$ where $\underline{\hat{U}_1}$ and $\underline{\hat{U}_2}$ are two estimators. The WMSE of this estimator is

$$\text{WMSE}(\underline{g}(\underline{\hat{T}^*})) = \alpha^2\text{WMSE}(\underline{g}(\underline{\hat{T}_1})) + (1-\alpha)^2\text{WMSE}(\underline{g}(\underline{\hat{T}_2})) + 2\alpha(1-\alpha)\text{WMXPE}(\underline{g}(\underline{\hat{T}_1},\underline{g}(\underline{\hat{T}_2})), \tag{26}$$

where

$$\text{WMXPE}(\underline{g}(\underline{\hat{T}_1},\underline{g}(\underline{\hat{T}_2})) = \sum_{ij} \omega_{ij}[\text{Cov}(\hat{U}_{1i},\hat{U}_{2j}) + \text{Bias}(\hat{U}_{1i})\text{Bias}(\hat{U}_{2j})] \tag{27}$$

is defined to be the Weighted Mean Cross-Product Error of $\underline{g}(\underline{\hat{T}_1})$ and $\underline{g}(\underline{\hat{T}_2})$. The WMSE of our composite estimator is minimized when

$$\alpha = \frac{\text{WMSE}(\underline{g}(\underline{\hat{T}_2})) - \text{WMXPE}(\underline{g}(\underline{\hat{T}_1}),\underline{g}(\underline{\hat{T}_2}))}{\text{WMSE}(\underline{g}(\underline{\hat{T}_1})) + \text{WMSE}(\underline{g}(\underline{\hat{T}_2})) - 2\text{WMXPE}(\underline{g}(\underline{\hat{T}_1}),\underline{g}(\underline{\hat{T}_2}))}. \tag{28}$$

To obtain an estimate of α, we substitute estimates of the WMSEs and the WMXPE into the above.

As an example of how this approach could be used, suppose a decision has been taken to adjust a provincial population total. To carry down the adjustment, we might consider using either across-the-board adjustment (*i.e.* adjust all sub-provincial quantities by the same factor), or a synthetic adjustment, where the adjustment is done separately within several age-sex groups. The across-the-board method has the advantage that it uses only the provincial estimate of undercoverage, which is likely to be more reliable than the estimates of undercoverage by age and sex at the province level. On the other hand, if undercoverage varies considerably among age and sex groups, and if the sub-provincial quantities indexed by i also differ in their age-sex composition, then the synthetic estimator may be better.

If estimates of the U_i are available from some source, then all covariance and bias components of the WMSEs and the WMXPE can be estimated (using formulae such as those in Table 3), and the optimum composite estimator involving the across the board and synthetic estimators can be estimated. Although for sub-provincial quantities the U_i will not usually exist in practice, the method can be investigated at higher levels. For example we could use the provinces as the quantities indexed by i and use across-the-board and synthetic adjustment factors computed at the Canada level. A second possibility is to construct an artificial population (*e.g.* as in Shirm and Preston (1987) or Wolter and Causey (1991)) where the U_i are assumed to be known.

5. FURTHER WORK

The results presented in this paper represent only a start to the investigation and comparison of the performance of various estimators of a set of population totals. There are several areas where considerable work is yet required.

First, further investigation of the WMSEs for the preliminary test and composite estimators in the more general cases described in Sections 3 and 4 is required. Although attempts to derive analytic expressions for these WMSEs have not yet been successful, the more general results for preliminary test estimators and Stein-rule estimators described by Judge and Bock (1978) may yet be found to apply. If so, this would help to answer questions such as: Can optimal critical values be found for the Fellegi-type preliminary test estimators of Sections 3.3 and 4.1? How does the WMSRE of the preliminary test estimator compare in practice to those of the other three estimators?

Second, more work is needed to explore the sensitivity of the results to different weightings in the loss function. The results of Section 3 were based on the use of a weight equal to the inverse of the census count or the adjusted census count for each province. If the provinces had been weighted differently, the results would change. A more general weight we might want to consider is $w_k = Y_k^{\gamma}$, where γ is some type of power parameter. The sensitivity of the results in Section 3 to various of values of γ could then be studied.

Finally, while the methods described in this paper provide a framework for developing and evaluating various estimators, the exact manner in which the methods will be applied has yet to be decided. Specific issues that must be resolved include:

1. What is the relative importance of different types of functions such as totals, shares and growth rates? Different functions give rise to different results, but in the end a single estimator must be chosen in order to maintain consistency.

2. At what geographic and demographic levels should these methods be applied? For example, should the preliminary test estimator or composite estimator described in Section 3 be applied at the province level, at the province by age group and sex level, or at even more detailed levels? The results obtained depend on the level of analysis used.

3. Could we even consider composite estimators for "high profile" estimators such as the provincial population totals? It might be difficult to explain to users why the adjustments do not coincide with the published estimates of undercoverage.

Because the resolution of issues such as these will require professional judgement, the decision about whether to adjust (and how to adjust) cannot be an automatic one based on completely pre-specified criteria. While the methods described in this paper can provide useful guidance, the final decision will require a careful balancing of the potential improvement in the accuracy of the estimates with consideration of how easily the methods can be communicated to and understood by users of the estimates program.

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