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Estimating Economic Cycles in Semi-Annual Series

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ABSTRACT

This paper presents a moving average which estimates the trend-cycle while eliminating seasonality from semi-annual series (observed twice yearly). The proposed average retains the power of all cycles which last three years or more; 90% of those of two years; and 55% of cycles of one year and a half. By comparison, the *two by two* moving average retains the power of respectively 75%, 50% and 25% of the same cycles.

KEY WORDS: Moving averages; Economic cycles; Spectral analysis.

1. INTRODUCTION

In some cases, semi-annual series exist for which there are no corresponding monthly data. In such instances, one cannot derive the seasonally adjusted semi-annual series from the monthly seasonally adjusted values. In addition, to our knowledge, there are no seasonal adjustment methods for semi-annual series.

This paper presents a moving average which eliminates seasonality and estimates the trend-cycle of semi-annual series. The approach of quadratic minimization used originates with Whittaker (1923) and was further developped by Leser (1961 and 1963), Cholette (1980), Schlicht (1981) and others.

The average derived has five terms and comprises a set of central weights for the semestres (half-years) at the centre of series; and two sets of end weights, for the two first and last semestres. Consequently there is no loss of estimates at the ends of series, as with the *two by four* moving average (used for quarterly series) for instance.

The spectral properties of the central weights prove to be superior to those of the *two by two* moving average, which first comes to mind as a way of processing semi-annual series. The properties of the end weights will also be examined.

2. ILLUSTRATION OF THE AVERAGE

Figure 1 shows the observed semi-annual original series z_t (dashed line) along with the trend-cycle c_t (solid line) estimated by the semi-annual cyclical average presented in this paper. As expected, the trend-cycle behaves smoothly and displays short run cycles, namely a three-year cycle extending from the second semestre (half-year) of 1977 to the first of 1980. An estimate is available for each observation, including the two first and last observations. The trend-cycle produced by the *two by two* moving average (dotted curve) on the other hand does not yield any estimate for the first and the last semestres. Furthermore, the two by two does not reach as deeply into the cyclical peaks and troughs compared to the proposed average.

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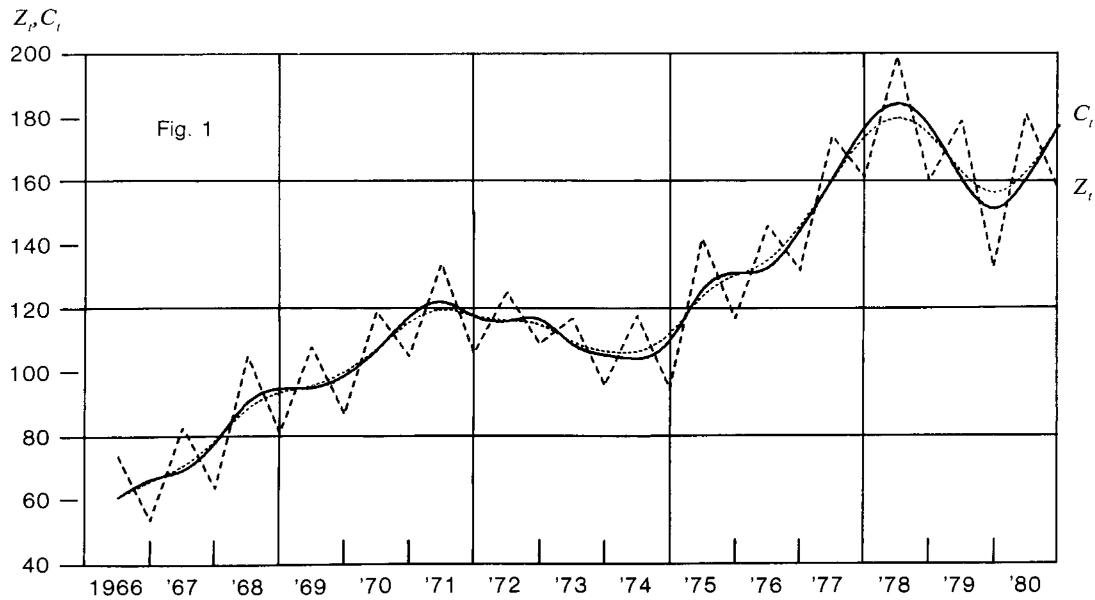

Figure 1. Semi-annual seasonal series (----) and its trend cycle (——) estimated by the proposed semi-annual cyclical moving average and by the *two by two* moving average (· · · ·)

3. WEIGHTS OF THE AVERAGE

Table 1-A displays the exact values of the weights of the semi-annual cyclical moving average used. The first row gives the *modified* central weights pertaining to estimates 3 to 28 (in Figure 1); the second, the end weights pertaining to the second-last estimate; and, the third, to the last estimate. Table 1-B shows the central weights derived according to the methodology of Section 6. We judged however these should be replaced by the *modified* central weights of Table 1-A for reasons to be explained.

Table 1-A

Exact weights of the proposed semi-annual cyclical average

Modified Central weights	−0.1000	0.2500	0.7000	0.2500	−0.1000
Second-last Set of weights	0.0625	−0.2500	0.3750	0.7500	0.0625
Last Set of weights	−0.0625	0.2500	−0.3750	0.2500	0.9375

Table 1-B

Unmodified central weights

−0.0625	0.2500	0.6250	0.2500	−0.0625

4. SPECTRAL ANALYSIS OF THE AVERAGE

The curves of Figures 2 and 3 represent the gain functions of the set of weights analysed. The gain value on the ordinate indicates the percentage of amplitude of the sinusoidal waves preserved, that is *passed* to the estimates, by the weights. The frequency of these waves is shown on the abscissa and varies from 0 to 0.500. Frequency 0.500 corresponds to the annual wave of two semestres (1/.50), that is to stable seasonality. Moving seasonality is accounted for by a few neighbouring quasi annual frequencies: 0.467 and 0.483.

Frequency 0.333 corresponds to a three (1/.33) semestre wave; frequency 0.250, four semestres; 0.200, to five semestres; 0.167, six semestre, etc. Frequencies associated with waves of three semestres or more (left of 0.333 in figures 2 and 3) pertain to the trend-cycle of series and constitute the target frequencies of the estimator.

The frequencies between 0.333 and 0.467 exclusively are associated with fluctuations of periodicity less than one and a half years and superior to the quasi-annual seasonal frequencies. They pertain to the irregular component of series. An ideal cyclical average should eliminate 100% of these irregular frequencies, 100% of the seasonal and quasi-seasonal frequencies and preserve only the cyclical frequencies from 0 to 0.333 inclusively.

a) Analysis of the central weights

The solid curve of Figure 2 shows that the modified central weights of the semi-annual cyclical average preserves 100% of all waves of five semestres (2 years) and more: everywhere left of frequency 0.200 the curve is above 100%. By comparison, the *two by two* moving average, represented by the dotted curve, only preservs 65% of the 5-semestre waves and 93% of the 10-semestres waves. Furthermore, the modified central weights pass 55% of 3-semestre waves and 90% of 4-semestre (2-year) waves; against 25% and 50% respectively for the two by two.

Both sets of weights completely eliminate stable seasonality, with gain valued at zero for the seasonal frequency 0.500; and nearly all the moving seasonality. However, the two by two eliminates slightly more of the irregular frequencies than the modified central weights. When choosing between the two averages, one then faces the following trade-off: to let the estimates contain more cyclical movements but also more irregularity or less cyclical movements and less irregularity. When a series is known to contain more cyclical movements (especially faster movements) than irregularity, the modifed central weights of the proposed average are certainly preferable to the two by two.

The dashed curve of Figure 2 represents the gain of the unmodified central weights of the semi-annual cyclical average, as obtained from Section 6. At the cyclical frequencies, its performance proves superior to that of the *two by two*; but, inferior to that of the modified central weights. For instance, the latter reproduces 101% of the 5-semestre waves (frequency 0.200) against 88% for the unmodified. The amplification of 5% (gain of 105%), at the 6-semestre (frequency 0.167) wave with the modified central weights, seems to us preferable to a comparable reduction of 6% (gain of 94%) with the unmodified set of weights. Indeed the analyst stands a better chance to detect an amplified signal in a series than a reduced signal.

b) Analysis of the end weights

Ideally, the gains of the end weights should be identical to the gain of the central weights. In such a case, end and central weights would have the same effect on the processed series (except for possible phase-shifts).

The gain of the weights for the second-last estimate (dotted line of Figure 3) is quite similar to the gain of the modified central weights (solid curve). Note that the former is more similar to the latter than to the set of unmodified central weights of Figure 2. This is the reason why we modified the weights.

The weights for the second-last estimate preserve the cyclical frequencies and eliminate stable seasonnality. However, they preserve 11 and 21% of the moving seasonality frequencies 0.467 and 0.483; and, even more of the noise frequencies. From the view point of the gain, the weights of the second-last estimate should yield less reliable estimates than the modified central weights.

Fig. 2

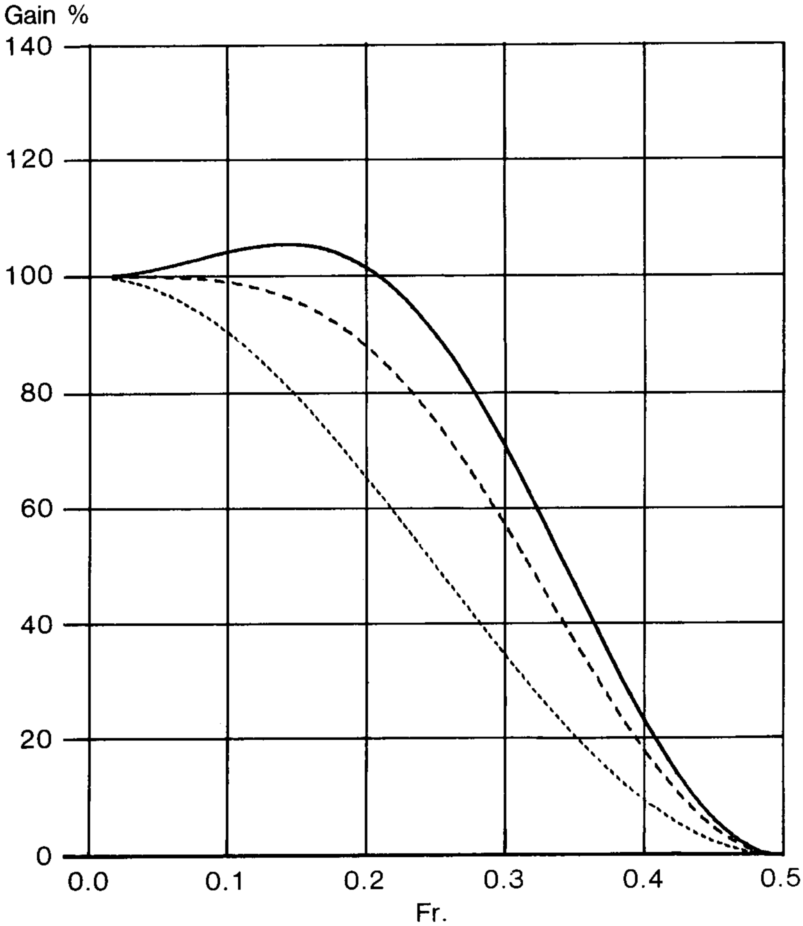

Figure 2. Gain functions of the modified (——) and non-modified (----) central weights of the proposed semi-annual cyclical average and of the *two by two* average (· · · ·)

The situation gets worse for the set of weights for the last estimate (broken line in Figure 3). Here, a strong amplification of some noise and fast cyclical frequencies is observed (gains reaching up to 137%). Caution should then be exrcised in interpreting the estimate yielded by these weights. One should perhaps disregard the last estimate completely, for series which are reputed to be irregular (containing those magnified frequencies).

As seen in Table 1, the end sets of weights are not symmetric. Consequently, they cause phase-shifts, which are compiled in number of semestres in Table 2 for certain selected frequencies. At the target cyclical frequencies, a small phase-shift is observed for the second-last weights. In this case, a cyclical wave of five semestres will be delayed by 0.09 semestres in the estimates; one of four semestres, by 0.16 semestres; and one of three semestres, by 0.28 semestres; etc.

The phase-shift reaches its maximum at the fundamental seasonal frequency (0.500). This does not matter however, since the frequency is totally eliminated by the weights. It does matter a little for the moving seasonality frequencies 0.467 and 0.483, since they are not completely eliminated.

Fig. 3

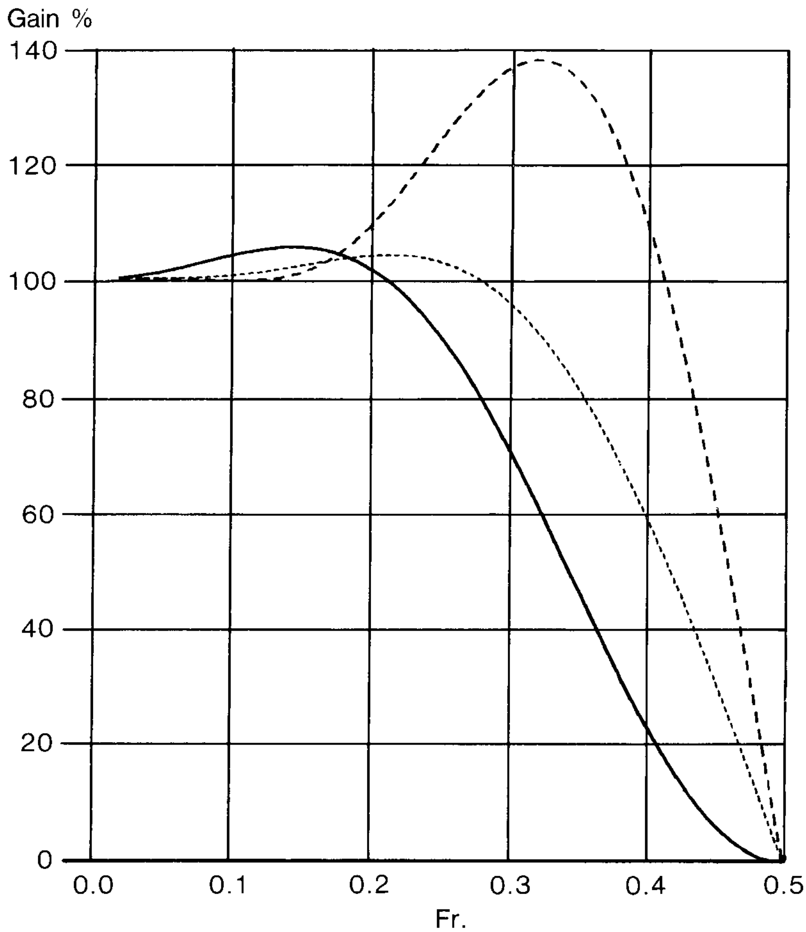

Figure 3. Gain functions of the modified central weights (——) and of the weights for the second-last (····) and the last (----) estimates

Table 2

Phase-shifts observed in number of semestres for the sets of end weights at certain selected frequencies

	second-last	last
cyclical frequencies:		
0.100 (10 semestres)	0.01	0.01
0.167 (6 semestres)	0.05	0.05
0.200 (5 semestres)	0.09	0.05
0.250 (4 semestres)	0.16	0.00
0.333 (3 semestres)	0.28	0.17
seasonal frequencies		
0.467	0.46	0.45
0.483	0.48	0.47
0.500 (2 semestres)	0.50	0.50

5. GRAPHICAL ANALYSIS OF END ESTIMATES

Figure 4 displays the preliminary estimates derived using the two sets of end weights for years 1968 to 1980, accompanied by the corresponding central final available estimates. Figure 4 a) shows the end estimates falling in the second semestre; and 4 a), in the first semestre. (One single plot would have been too crowded.)

If the central estimates are considered as true (or at least more reliable, the end estimates are seen to cause five false signals: in 1968 (arrow in fig. 4 b), in 1972 (4 a), in 1974 (4 b), in 1975 (4 a) and in 1976 (4 b). A false signal is said to occur here when the end estimates show a change in the direction of the trend-cycle and when that change is later contradicted by the central final estimates (becoming available with new observations). These false signals tend to appear when the series slows down in one direction and resumes the movement in the same direction. When there is a strong change of direction like in 1978, this does not seem to occur.

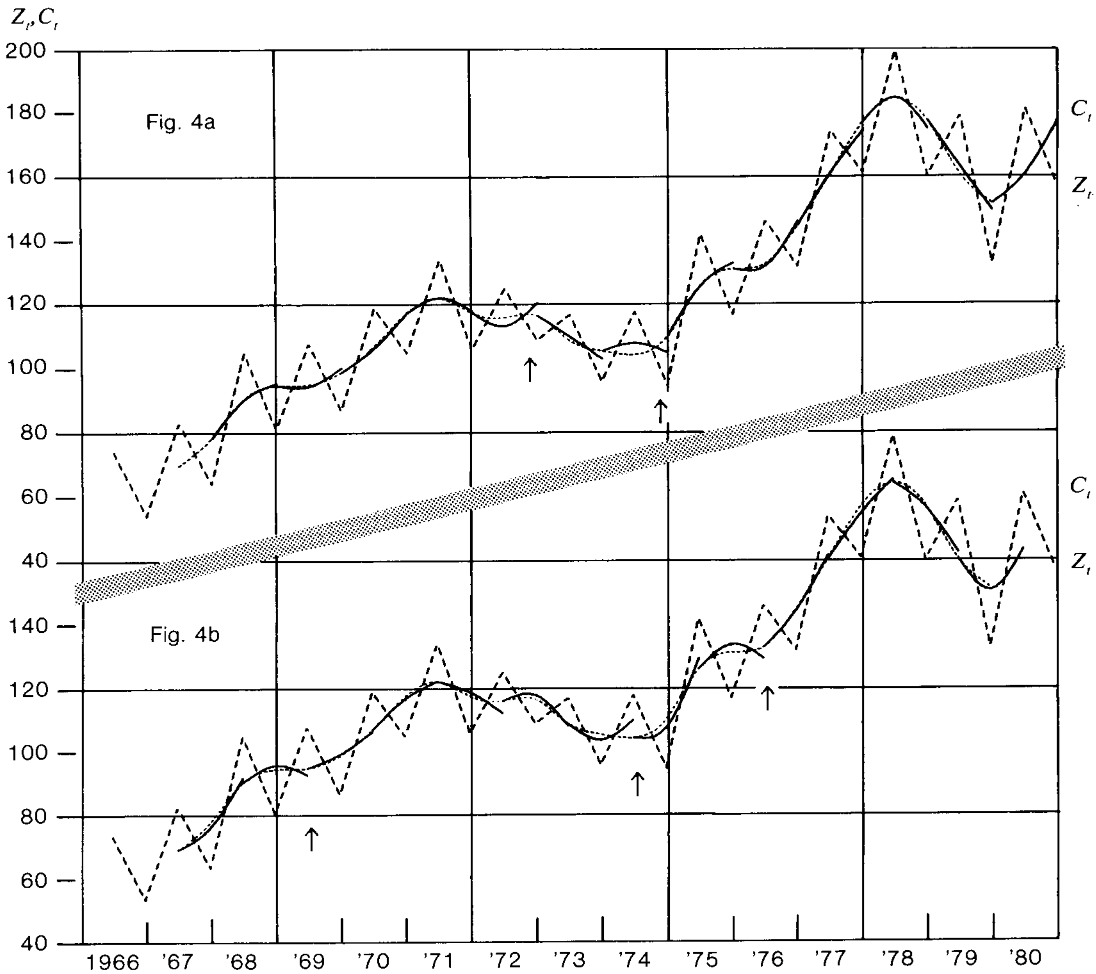

Figure 4. Semi-annual seasonal series (----); preliminary estimates of its trend-cycle by the ends weights (——) of the proposed semi-annual cyclical moving average a) for the second semestres and b) for the first semestres; final estimates (····) by the central weights of the average.

If the estimates derived by the last set of weights were omitted, many false signals would *disappear.* However, the estimates sould become less timely. This illustrates the statistician's dilemma between the timelyness and the reliability of estimates under any estimation method. (In practise, a serious analyst would wait for at least one confirmation of a signal before *believing* it).

Apart from the five false signals mentionned, the preliminary estimates display a movement which is very similar and sometimes undistinguishable from that of the final estimates.

5. CALCULATION OF THE WEIGHTS OF THE SEMI-ANNUAL CYCLICAL AVERAGE

The observed series z_t comprises the trend-cycle c_t to be estimated and a seasonal-irregular residual $s_t + e_t$ $(= z_t - c_t)$:

$$z_t = c_t + (s_t + e_t), \ t = 1, \ldots, 5. \tag{1}$$

Following the approach of Leser (1961 and 1963) and of Cholette (1980), the desired trend-cycle minimizes the quadratic sum of fourth differences (first term of (2)). On the five-semestre estimation interval, the component as much as possible approximates a time polynomial of the third degree. This specification allows for a full economic cycle with its four phases of expansion, turning-point, recession and recovery over the interval.

The seasonal-irregular residual $(z_t - c_t)$ minimizes the quadratic sum of first seasonal differences taken on corresponding semestres (second term of (2)). This specification means that the seasonal-irregular residual of one semestre should resemble that of the same semestre in the neighbouring year as much as possible.

Furthermore, the seasonal-irregular residuals minimize the quadratic sum of their sums on two consecutive semestres (third term of (2)). This criterion indicates that the seasonality of two neighbouring semestres should cancel out and that the irregularity should not affect the level of the desired trend-cycle.

The three criteria specified for the components combine into the following objective function:

$$\begin{aligned} f(c) &= \sum_{t=5}^{5} (c_t - 4c_{t-1} + 6c_{t-2} - 4c_{t-3} + c_{t-4})^2 \\ &+ \sum_{t=3}^{5} \{(z_t - c_t) - (z_{t-2})\}^2 + \sum_{t=2}^{5} \{(z_t - c_t) + (z_{t-1} - c_{t-1})\}^2 \end{aligned} \tag{2}$$

Equation (3) can be rewritten in linear algebra:

$$\begin{aligned} f(C) &= C'A'AC + (Z - C)'B'B(Z - C) + (Z - C)'F'F(Z - C) \\ &= C'HC + (C - Z)'G(C - Z) \end{aligned} \tag{3}$$

where A, B and C respectively stand for the matrix operators of quadruple differences, first seasonal differences and annual sums defined as follows:

$$A = [\,1 \ \ -4 \ \ 6 \ \ -4 \ \ 1\,], \quad B = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The normal equations associated with (3) read

$$dF/dC = 2HC + 2G(C - Z) = 0 \tag{4}$$

and imply solution:

$$C = (H + G)^{-1}GZ = WZ \tag{5}$$

The third central row of matrix W contain the non-modified central weights of the semi-annual cyclical average ot Table 1-B; and the fourth and fifth row, the end weights of rows 2 and 3 of Table 1-A.

7. HISTORY OF MOVING AVERAGES BY QUADRATIC MINIMIZATION

This approach of quadratic minimization originates with Whittaker (1923). Leser (1961, 63) showed how quadratic minimization could be applied to develop cyclical moving averages. Cholette (1980) proposed substitutes for the two by twelve and the two by four moving averages. These substitutes were incorporated into the Dagum (1980) seasonal adjustment programme as optional.

The semi-annual cyclical average presented in this paper could also be incorporated in a seasonal adjustment method of the X-11 type. This would allow seasonally adjusting semi-annual series and the calculation of the seasonal factors by means of the seasonal moving averages usually applied for monthly and quarterly series.

8. CONCLUSION

This paper presented a 5-term moving average which eliminates seasonality from semi-annual time series. The estimator reproduces the economic cycles more exactly than the two by two moving average. The two by two also has the disadvantage of not providing any estimate for the first and last semestres of the series.

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